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Generalized Berezin-Toeplitz quantization of Kähler supermanifolds

Calin Iuliu-Lazaroiu, Daniel McNamee and Christian Sämann

Trinity College Dublin,
Dublin 2, Ireland

E-mail: calin@maths.tcd.ie, danmc@maths.tcd.ie, saemann@maths.tcd.ie

Abstract: We extend the construction of generalized Berezin and Berezin-Toeplitz quantization to the case of compact Hodge supermanifolds. Our approach is based on certain super-analogues of Rawnsley's coherent states. As applications, we discuss the quantization of affine and projective superspaces. Furthermore, we propose a definition of supersymmetric sigma-models on quantized Hodge supermanifolds. The corresponding quantum field theories are finite and thus yield supersymmetry-preserving regularizations for QFTs defined on flat superspace.

Keywords: Superspaces, Non-Commutative Geometry, Differential and Algebraic Geometry

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Contents

1 Motivation and context

Geometric quantization [1] and its variants (such as Berezin and Berezin-Toeplitz quantization) arose as techniques for 'quantizing' symplectic manifolds in an attempt to give rigorous formulations to the canonical quantization procedure for systems with a finite number of degrees of freedom. Beyond its importance in physics, this theory leads to certain notions of 'quantum' symplectic geometry, which are of independent mathematical interest. It also leads to a certain class of regularizations for quantum field theories [2]. Finally, the worldvolumes of D-branes placed in certain superstring vacua can be described in terms of quantized spaces [3, 4].

Since many classical mechanical models admit canonical formulations containing both even and odd variables, it is natural to extend such quantization prescriptions to the case of symplectic supermanifolds. In particular, quantized Hodge supermanifolds should provide supersymmetry-preserving regularizations of supersymmetric quantum field theories.

Geometric quantization was independently developed by Kostant [5, 6] and Souriau [7] and proceeds in two steps. First, one fixes a positive complex line bundle over the manifold to be quantized, whose space of L^2-sections yields a *prequantization*. Second, one picks a polarization on this bundle, which can be used in order to reduce the space of L^2-sections to a proper subspace, thereafter identified with the quantum Hilbert space. The question of prequantization for supermanifolds was considered in [6], later expanded on in [8]. The appropriate definition of positive super line bundles was given in [9].

In the present paper, we do not consider geometric quantization but the closely related Berezin and Berezin-Toeplitz methods, extending them to Hodge supermanifolds. More precisely, we consider the superextension of the generalized Berezin and Berezin-Toeplitz quantizations of [10], which subsume the classical Berezin and Berezin-Toeplitz cases. Some previous work in this direction, though restricted to the case of certain homogeneous supermanifolds, can be found[1] in [11–13].

Berezin quantization was introduced in [15] while its Berezin-Toeplitz variant is discussed e.g. in [16] and more recently in [17]. Generalized Berezin and Berezin-Toeplitz quantizations were defined and analyzed in [10] and include interesting new possibilities, such as *Berezin-Bergman quantization*. The latter prescription is natural in contexts arising from algebraic geometry. In the present paper, we extend the results of [10] to the class of Hodge supermanifolds.

The paper is organized as follows. Section 2 recalls some basic notions of supergeometry and introduces the concept of Hodge supermanifolds. In section 3, we give the construction of supercoherent states and of generalized Berezin and Berezin-Toeplitz quantizations for Hodge supermanifolds, as well as a brief discussion of their properties. Section 4 considers various special cases: the analogues of classical Berezin and Berezin-Toeplitz quantizations, the quantizations of affine and projective superspaces as well as Berezin-Bergman superquantization. The last section shows how one can employ our methods to construct supersymmetry-preserving regularizations of supersymmetric quantum field theories.

[1] The work cited relies on group theoretic methods and on a super-generalization of Perelomov's coherent states [14], none of which can can be applied directly to general Hodge supermanifolds as defined below.

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2 Hodge supermanifolds, polarizations and Bergman supermetrics

In this section, we recall some basic notions from the theory of supermanifolds. The reader can consult [18, 19] for further details. To understand some of the concepts presented in the following, the reader might find it helpful to first study the corresponding definitions for ordinary manifolds as given in [10].

2.1 Super hermitian pairings

Recall that a complex supervector space E is a vector space over the complex numbers endowed with a $\mathbb{Z}_2$-grading $E = E_+ \oplus E_-$. A *super Hermitian pairing* on E is a $\mathbb{C}$-sesquilinear even form $(\ ,\ ) : E \times E \to \mathbb{C}$ which is graded-Hermitian, i.e. it satisfies the condition

$$(s,t) = (-1)^{\tilde{s}\tilde{t}}\overline{(t,s)} \tag{2.1}$$

for any two $\mathbb{Z}_2$-homogeneous elements s, t of E with degrees $\tilde{s}, \tilde{t}$. Our convention for sesquilinear forms is that they are antilinear in the *first* variable.

Evenness of the pairing implies that (s,t) vanishes unless $\tilde{s} = \tilde{t}$. Hence a super Hermitian pairing is completely determined by its restrictions to E_+ and E_-. Relation (2.1) shows that the first restriction is a Hermitian form on E_+, while the second is anti-Hermitian on E_-. Thus a super Hermitian form can be expressed as:

$$(s,t) = (s_+, t_+)_+ + i(s_-, t_-)_- \tag{2.2}$$

where $(\ ,\ )_\pm$ are Hermitian pairings on $E_\pm$. Here $s = s_+ + s_-$ and $t = t_+ + t_-$ are the decompositions of s, t into even and odd components. Conversely, a choice of Hermitian forms on $E_\pm$ determines a super Hermitian pairing on E.

A super Hermitian pairing $(\ ,\ )$ is called *nondegenerate* if it is nondegenerate as a sesquilinear form, i.e. if vanishing of (s,t) for all t implies $s = 0$. This amounts to the requirement that $(\ ,\ )_+$ and $(\ ,\ )_-$ are both nondegenerate. A super Hermitian pairing is called a *superscalar product* if it is nondegenerate and if $(\ ,\ )_+$ is positive-definite on E_+.

A super Hermitian form on E induces an even antilinear map $\Phi : E \to E^* = \underline{\mathrm{Hom}}_{\mathbb{C}}(E, \mathbb{C})$ given by:

$$\Phi(s)(t) = (s,t) \ ,$$

which is bijective iff the pairing is nondegenerate. In that case, the dual supervector space E^* has an induced super Hermitian pairing $(\ ,\ )_*$ given by

$$(\eta, \rho)_* = (-1)^{\tilde{\eta}\tilde{\rho}}(\Phi^{-1}(\rho), \Phi^{-1}(\eta)) \quad . \tag{2.3}$$

Let us fix a nondegenerate super Hermitian pairing on E. The *super Hermitian conjugate* of a homogeneous linear operator A on E is defined through:

$$(As, t) = (-1)^{\tilde{A}\tilde{s}}(s, A^\dagger t) \quad \forall s, t \in E \ \text{ homogeneous} \tag{2.4}$$

and extended to inhomogeneous operators in the obvious manner. When A is even ($A = A_+ + A_-$ with $A_\pm \in \mathrm{End}(E_\pm)$), this boils down to $A^\dagger = A_+^\dagger \oplus A_-^\dagger$, where $A_\pm^\dagger : E_\pm \to E_\pm$

are the Hermitian conjugates of $A_\pm$ with respect to $(\ ,\ )_\pm$. When A is odd ($A = A_1 + A_2$ with $A_1 : E_- \to E_+$ and $A_2 : E_+ \to E_-$), we find $A^\dagger = i(A_2^\dagger + A_1^\dagger)$ i.e. $(A^\dagger)_1 = iA_2^\dagger$ and $(A^\dagger)_2 = iA_1^\dagger$, where $A_1^\dagger : E_+ \to E_-$ and $A_2^\dagger : E_- \to E_+$ are the Hermitian conjugates of A_1 and A_2 with respect to the pairings $(\ ,\ )_+$ and $(\ ,\ )_-$ on E_+ and E_-.

Super Hermitian conjugation gives a conjugation of the superalgebra $(\underline{\mathrm{End}}(E), \circ)$, i.e. an even and involutive antilinear antiautomorphism of this superalgebra. In particular, we have:

$$(AB)^\dagger = (-1)^{\tilde{A}\tilde{B}} B^\dagger A^\dagger \quad . \tag{2.5}$$

This superalgebra is also endowed with the usual supertrace $\mathrm{str} : \underline{\mathrm{End}}(E) \to \mathbb{C}$, which is an even map and satisfies:

$$\mathrm{str}(AB) = (-1)^{\tilde{A}\tilde{B}} \mathrm{str}(BA) \quad . \tag{2.6}$$

Notice[2] that $\mathrm{str}(A^\dagger) = \overline{\mathrm{str}(A)}$.

The underlying supervector space $\underline{\mathrm{End}}(E)$ carries the *super Hilbert-Schmidt pairing* induced by $(\ ,\ )$, which is defined through:

$$\langle A, B\rangle_{\mathrm{HS}} = \mathrm{str}(A^\dagger B) \in \mathbb{C} \quad . \tag{2.7}$$

This is itself a non-degenerate super Hermitian pairing on $\underline{\mathrm{End}}(E)$, and in particular it satisfies:

$$\overline{\langle A, B\rangle_{\mathrm{HS}}} = (-1)^{\tilde{A}\tilde{B}} \langle B, A\rangle_{\mathrm{HS}} \quad . \tag{2.8}$$

Notice that $\langle\ ,\ \rangle_{\mathrm{HS}}$ need not be a superscalar product even when $(\ ,\ )$ is.

2.2 Supermanifolds

Throughout this paper we will work with supermanifolds in the sense of Berezin (see [18]). Recall that a *superspace* over $\mathbb{C}$ is a locally super ringed space over the complex numbers, i.e. a pair $(X, \mathcal{A})$ where X is a topological space and $\mathcal{A}$ is a sheaf of superalgebras over $\mathbb{C}$ such that the stalk $\mathcal{A}_x$ of $\mathcal{A}$ at any point $x \in X$ is a local superalgebra. Given a superspace, we let $\mathcal{A}_n \subset \mathcal{A}$ be the subsheaf of nilpotent elements of $\mathcal{A}$, and set $\mathcal{A}_{\mathrm{red}} := \mathcal{A}/\mathcal{A}_n$ and $\hat{\mathcal{A}} = \mathcal{A}_n/\mathcal{A}_n^2$. We say that a superspace $(X, \mathcal{A})$ is a real (resp. complex) *supermanifold* of dimension $(m|n)$ if:

(1) $(X, \mathcal{A}_{\mathrm{red}})$ is the locally ringed space of smooth (resp. holomorphic) complex-valued functions associated with a real (resp. complex) manifold structure on X of real (resp. complex) dimension m,

(2) $\hat{\mathcal{A}}$ is locally free of purely odd finite rank $0|n$ as a sheaf of $\mathcal{A}_{\mathrm{red}}$-supermodules,

(3) $\mathcal{A}$ and $\wedge^n_{\mathcal{A}_{\mathrm{red}}} \hat{\mathcal{A}}$ are *locally* isomorphic as sheaves of superalgebras over $\mathcal{A}_{\mathrm{red}}$.

[2] It suffices to check this for even operators A since $\mathrm{str}(A)$ vanishes when A is odd.

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The natural surjection $\mathcal{A} \to \mathcal{A}_{\mathrm{red}}$ induces a ringed space embedding $(X, \mathcal{A}_{\mathrm{red}}) \to (X, \mathcal{A})$ whose underlying map of spaces is the identity on points of X. The ringed space $(X, \mathcal{A}_{\mathrm{red}})$ is denoted by X_{red} and called the *reduced* space associated with $(X, \mathcal{A})$; this will also be identified with the corresponding (real or complex) manifold. According to (1) in the definition, X_{red} is the ringed space of smooth (resp. holomorphic) functions associated with a real (resp. complex) manifold of real (resp. complex) dimension m. In the real case, we have $\mathcal{A}_{\mathrm{red}} = \mathcal{C}^\infty(X_{\mathrm{red}})$ while in the complex case we have $\mathcal{A}_{\mathrm{red}} = \mathcal{O}(X_{\mathrm{red}})$. A supermanifold X is called compact, connected etc. if the underlying manifold X_{red} has the corresponding property.

Each of the local rings $\mathcal{A}_x$ $(x \in X)$ is an augmented superalgebra, whose augmentation morphism is the natural projection $\epsilon_x : \mathcal{A}_x \to \mathcal{A}_x/\mathfrak{m}_x = k$ ($\mathfrak{m}_x$ is the unique maximal ideal of $\mathcal{A}_x$ while $k = \mathbb{R}$ or $\mathbb{C}$ for real and complex supermanifolds, respectively). This is a k-superalgebra morphism from $\mathcal{A}_x$ to k, where the latter is viewed as a commutative superalgebra over itself concentrated in degree zero. The augmentation morphism is sometimes called the 'body map', while $\epsilon_x(f)$ is called the 'body' of an element f of $\mathcal{A}_x$. When X has dimension $(m|n)$, we have isomorphisms of superalgebras $\mathcal{A}_x \cong k[\zeta^1 \ldots \zeta^n]$ (the Grassmann k-algebra on n odd generators $\zeta^1 \ldots \zeta^n$) for any $x \in X$. Furthermore, $\mathfrak{m}_x$ can be identified with the maximal ideal $\langle \zeta^1 \ldots \zeta^n \rangle$ of this Grassmann superalgebra.

Condition (2) in the definition means that $\hat{\mathcal{A}}$ is the sheaf of smooth (resp. holomorphic) sections of the parity change ΠE of a complex rank n vector bundle E over X_{red}, which is a holomorphic bundle in the complex supermanifold case. By convention, we will denote $\mathcal{A}$ by $\underline{\mathcal{O}}$ respectively $\underline{\mathcal{C}}$ for the case of complex resp. real supermanifolds, and let $\mathcal{O}$ respectively $\mathcal{C}$ denote the corresponding reduced sheaves.

The following notations will be used later in this paper. For any point $x \in X$, we let $\mathcal{A}_x^\times$ denote the subgroup of invertible elements $\mathcal{A}_x$, i.e. those elements f of $\mathcal{A}_x$ such that $\epsilon_x(f) \neq 0$. We also let $\mathcal{A}_x^{\times,ev}$ denote the subgroup consisting of all even elements of $\mathcal{A}_x^\times$. Finally, we let $\mathcal{A}^\times$ and $\mathcal{A}^{\times,ev}$ be the subsheaves of $\mathcal{A}$ consisting of those elements whose stalk values at all points x belong to $\mathcal{A}_x^\times$ and $\mathcal{A}_x^{\times,ev}$ respectively. We have sheaf inclusions $\mathcal{A}^{\times,ev} \subset \mathcal{A}^\times \subset \mathcal{A}$. When X is a real supermanifold, we let $\mathcal{C}_{>0}$ denote the subsheaf of $\mathcal{C}(X)$ consisting of 'superfunctions with positive body'. More precisely, we set $\mathcal{C}_{>0}(U) = \{f \in \mathcal{C}(U) | \epsilon_x(f(x)) > 0 \quad \forall x \in U\}$, where U is any open subset of X (here $f(x) \in \mathcal{C}_x$ is the stalk value of f at x). Notice that $\mathcal{C}_{>0}$ is a subsheaf of $\mathcal{C}^\times$.

Supervector bundles. Let $(X, \mathcal{A})$ be a supermanifold of dimension $(m|n)$ over $k = \mathbb{R}$ or $\mathbb{C}$ (i.e. a real or complex supermanifold). A superfibration $E \xrightarrow{\pi} X$ is a fibration in the category of supermanifolds over k, while a super fiber bundle is a fiber bundle in that category. Such a fiber bundle is called a supervector bundle of rank $(p|q)$ if its local trivializations over sufficiently small sets are modeled on the bundle $U \times \mathbb{A}^{p|q}$ with $U \subset X$, while its transition functions are valued in the supergroup $\mathrm{GL}_k(p|q)$. Here $\mathbb{A}^{p|q}$ is the affine superspace of dimension $(p|q)$ over k. The associated sheaf of sections is defined through $\mathcal{E} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}_{E,\mathrm{lin}})$, where $\mathcal{A}_{E,\mathrm{lin}}$ is the subsheaf of the structure supersheaf $\mathcal{A}_E$ of E whose local sections are linear along the fibers of E. This sheaf is locally free of rank $(p|q)$ as a sheaf of $\mathcal{A}$-supermodules. Conversely, any sheaf $\mathcal{E}$ of $\mathcal{A}$-supermodules which is locally

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free of rank $(p|q)$ can be viewed as the sheaf of sections of a supervector bundle of rank $(p|q)$ given by $E = \mathrm{Spec}[S^\bullet_{\mathcal{A}}(\mathcal{E}^{\mathrm{v}})]$ (with the obvious projection) where $\mathcal{E}^{\mathrm{v}} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ is the dual sheaf and $S^\bullet_{\mathcal{A}}$ is the functor on the category of sheaves of $\mathcal{A}$-supermodules induced by taking the total graded symmetric algebra over a supermodule. Connections on super-vector bundles are defined by mimicking the classical theory.

Restriction and reduction of sheaves and supervector bundles. Given a sheaf $\mathcal{E}$ of $\mathcal{A}$-supermodules on X, its *restriction to* X_{red} is the sheaf on X defined through:

$$\mathcal{E}_{\mathrm{res}} := \mathcal{A} \otimes_{\mathcal{A}_{\mathrm{red}}} \mathcal{E} \cong \mathcal{E}/\mathcal{A}_n \cdot \mathcal{E} \quad . \tag{2.9}$$

This is a sheaf of $\mathcal{A}_{\mathrm{red}}$-supermodules, i.e. sheaf of supermodules on the ringed space X_{red}. The subsheaf $\mathcal{E}_{\mathrm{red}} := (\mathcal{E})^+_{\mathrm{res}}$ of even elements in $\mathcal{E}_{\mathrm{res}}$ is called the *reduction* of $\mathcal{E}$; it is an ordinary sheaf of modules over X_{red}.

When $\mathcal{E}$ is locally free of rank $(p|q)$ with associated supervector bundle $E \xrightarrow{\pi} X$, then $\mathcal{E}$ is locally isomorphic with the free sheaf $\mathcal{A}^{p|q}$ and $\mathcal{E}_{\mathrm{res}}$ is locally isomorphic with $\mathcal{A}^{p|q}_{\mathrm{red}} = \mathcal{A}^{\oplus p}_{\mathrm{red}} \oplus (\Pi\mathcal{A}_{\mathrm{red}})^{\oplus q}$, thus locally free as a sheaf of $\mathcal{A}_{\mathrm{red}}$-supermodules and represented by a *super*vector bundle $E_{\mathrm{res}} = E^+_{\mathrm{res}} \oplus E^-_{\mathrm{res}} \xrightarrow{\pi_{\mathrm{res}}} X_{\mathrm{red}}$ of rank (p,q) over X_{red}, called the restriction of the supervector bundle E. The image of a global section s of E though the projection $\mathcal{E}(X) \to \mathcal{E}_{\mathrm{res}}(X)$ is denoted by s_{res} and called the *restriction* of s. Notice that $s_{\mathrm{res}} = 1_{\mathcal{A}(X)} \otimes_{\mathcal{A}_{\mathrm{red}}(X)} s$. The sheaf $\mathcal{E}_{\mathrm{red}}$ is again locally free and represented by the ordinary vector bundle $E_{\mathrm{red}} = E^+_{\mathrm{res}} \to X_{\mathrm{red}}$ (the even subbundle of E_{res}) on X_{red}. Notice that the total space E_{red} is the underlying ordinary manifold of the total supermanifold E which is the total space of $E \xrightarrow{\pi} X$. Also notice that when E has rank $(p|0)$ (i.e. when $q = 0$), then $E_{\mathrm{red}} = E_{\mathrm{res}}$.

Notice that the sheaf of supersections of E can be described as $\mathcal{E} = \mathcal{A} \otimes_{\mathcal{A}_{\mathrm{red}}} \mathcal{E}_{\mathrm{res}}$.

Natural sheaves and bundles. The *tangent sheaf* of X is the sheaf $\mathcal{T}_X := \underline{\mathrm{Der}}(\mathcal{A})$ of derivations of $\mathcal{A}$. This is locally free of rank $(m|n)$. The super vector bundle TX associated with $\mathcal{T}_X$ is called the *tangent bundle* of X. Super-vector fields on X are defined as global supersections of $\mathcal{T}_X$. The *cotangent sheaf* is the dual sheaf $\mathcal{T}_X^{\mathrm{v}} = \mathrm{Hom}_{\underline{\mathcal{O}}}(\mathcal{T}_X, \mathbb{C})$, whose global supersections are the one-forms on X. It is again locally free and represented by the *cotangent bundle* T^*X. Similarly, one defines the supertensor sheaves $\mathcal{T}_X\binom{p}{q} = \mathcal{T}_X^{\otimes p} \otimes (\mathcal{T}_X^{\mathrm{v}})^{\otimes q}$, whose global sections are tensor superfields of type $\binom{p}{q}$ on X. These sheaves are locally free and represented by the tensor bundles $\mathcal{T}\binom{p}{q}(X)$. The (locally free) sheaf of p-forms is $\Omega^p_X := \wedge^p \mathcal{T}^{\mathrm{v}}_X$, where $\wedge$ is the graded wedge product; this sheaf is represented by the p-form bundle $\Lambda^p T^*M$ (generally, one has $\Omega^p_X \neq 0$ for all $p \geq 0$). The de Rham super differential d is defined by mimicking the classical construction. We also have the symmetric supertensor sheaves $S^p_{\mathcal{A}}(\mathcal{T}_X)$, which are represented by the vector super bundle $S^p(TX)$ etc. The reductions of all these sheaves and bundles are the corresponding natural sheaves and bundles of X_{red}. For example, we have $(TX)_{\mathrm{red}} = T(X_{\mathrm{red}})$ etc.

2.3 Complex supermanifolds

A complex supermanifold $(X, \underline{\mathcal{O}})$ of dimension $(m|n)$ has an underlying real supermanifold $(X, \underline{\mathcal{C}})$ of dimension $(2m|2n)$ (see [20]). We have a morphism of ringed spaces $(X, \underline{\mathcal{C}}) \to$

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$(X, \underline{\mathcal{O}})$ whose underlying map of spaces is the identity and whose sheaf map $\underline{\mathcal{O}} \to \underline{\mathcal{C}}$ is an inclusion. There is a local isomorphism $\underline{\mathcal{C}} \cong \bar{\underline{\mathcal{O}}} \otimes_{\mathcal{C}} \underline{\mathcal{O}}$ where the sheaf $\bar{\underline{\mathcal{O}}}$ of antiholomorphic superfunctions and the conjugation $^{-} : \underline{\mathcal{O}} \to \bar{\underline{\mathcal{O}}}$ are defined [20] using the fact that $\underline{\mathcal{O}}$ is locally the exterior algebra of an $\mathcal{O}$-supermodule. Similarly, we have a local isomorphism $\hat{\underline{\mathcal{C}}} \cong \bar{\hat{\underline{\mathcal{O}}}} \otimes_{\mathcal{C}} \hat{\underline{\mathcal{O}}}$. As for ordinary complex manifolds, we have super-Dolbeault decompositions:

$$\Omega_X^k = \oplus_{p+q=k} \Omega_X^{p,q} \ ,$$

and the global sections of $\Omega_X^{p,q}$ are called (p,q)-forms on X. Decomposing d accordingly gives the Dolbeault super differentials ∂ and $\bar{\partial}$.

For any holomorphic supervector bundle E of rank $(p|q)$ on $(X, \underline{\mathcal{O}})$, we let $\underline{\mathcal{O}}(E)$ denote the sheaf of holomorphic supersections of E and $\underline{\mathcal{C}}(E)$ its sheaf of smooth supersections. These sheaves are locally free of rank $(p|q)$ respectively $(2p|2q)$ over $\underline{\mathcal{O}}$ and $\underline{\mathcal{C}}$ respectively. We have $\underline{\mathcal{C}}(E) = \underline{\mathcal{C}} \otimes_{\mathcal{C}^\infty} \mathcal{C}^\infty(E_{\rm res})$ and $\underline{\mathcal{O}}(E) = \underline{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}(E_{\rm res})$, where $\mathcal{C}(E_{\rm res})$ and $\mathcal{O}(E_{\rm res})$ are the ordinary sheaves of smooth resp. holomorphic sections of the supervector bundle $E_{\rm res}$ over the ordinary manifold $X_{\rm red}$.

We let $H^0(E) = \underline{\mathcal{O}}(E)(X)$ and $\Gamma(E) = \underline{\mathcal{C}}(E)(X)$ denote the spaces of global holomorphic and smooth supersections; these are supermodules over the superalgebras $\underline{\mathcal{O}}(X)$ and $\underline{\mathcal{C}}(X)$ respectively. When X is compact, we have $\mathcal{O}(X) = \mathbb{C}$ while $\mathcal{C}(X)$ is infinite-dimensional as a $\mathbb{C}$-vector space unless $X_{\rm red}$ consists of a finite set of points. In this case, $H^0(E)$ is a finite-dimensional vector space while $\Gamma(E)$ is infinite-dimensional as a vector space unless $X_{\rm red}$ consists of a finite set of points.

A *holomorphic super line bundle* on $(X, \underline{\mathcal{O}})$ is a holomorphic supervector bundle L of rank $(1|0)$. We say that L is *positive* if $L_{\rm red}$ is positive as an ordinary line bundle over X. Notice that any super line bundle satisfies $L_{\rm res} = L_{\rm red}$.

Hermitian structures. Let E be a complex supervector bundle and $\mathcal{E} = \mathcal{C}(E)$ be its sheaf of smooth supersections. A *super Hermitian pairing* h on E is a global supersection of the sheaf $\mathrm{Hom}_{\underline{\mathcal{C}}}(\overline{\mathcal{E}} \otimes_{\underline{\mathcal{C}}} \mathcal{E}, \underline{\mathcal{C}})$ such that its value h_p on the stalk at any point $p \in X$ is a super Hermitian pairing on the fiber E_p. We say that h is nondegenerate if each h_p is. We say that h is positive-definite (or a Hermitian supermetric) if each h_p is positive-definite, i.e. if $h_{\rm red}$ is a Hermitian *metric* on the bundle $E_{\rm red}$. A Hermitian supermetric g on TX is called a Hermitian supermetric on X, in which case the pair (X, g) is called a Hermitian supermanifold (in this case, $(X_{\rm red}, g_{\rm red})$ is a Hermitian manifold).

Kähler supermanifolds. A *Kähler super form* on X is a nondegenerate $(1,1)$-form ω such that $d\omega = 0$ and such that $\omega_{\rm red}$ is positive-definite. By nondegeneracy, we mean that the stalk values ω_p are nondegenerate bilinear pairings on the vector superspaces $T_pX = \mathcal{T}_{X,p}$ for each $p \in X$. A *Kähler supermanifold* is a Hermitian supermanifold (X, g) such that the 2-form $\omega_g := i\bar{\partial}\partial g$ is a Kähler super form.

Projective superspaces. Let $V = V_+ \oplus V_-$ be a complex supervector space with $\dim_{\mathbb{C}} V_+ = m+1$ and $\dim_{\mathbb{C}} V_- = n$. The projectivisation of V is the projective superspace $\mathbb{P}V = (\mathbb{P}V_+, \wedge^\bullet[V_- \otimes O_{\mathbb{P}V_+}(-1)])$, viewed as a (split[3]) complex supermanifold. This comes endowed with a holomorphic super line bundle $H := O(1)$ called the hyperplane bundle, whose powers we denote by $O(k) := O(1)^{\otimes k}$. The dual holomorphic superbundle $O(-1) = O(1)^*$ is called the tautological super line bundle. A $\mathbb{Z}_2$-homogeneous basis $e_0 \dots e_{m+n}$ of V (with $e_0 \dots e_m \in V_+$ and $e_{m+1} \dots e_n \in V_-$) determines supercoordinates on the affine supermanifold $\mathbb{A}_V$ associated with V, which in turn give a basis of the space of global holomorphic supersections $z_0 \dots z_{m+n}$ of $O(1)$. The latter are the *homogeneous supercoordinates* of $\mathbb{P}V$ determined by the given homogeneous basis of V. The homogeneous supercoordinate ring $\oplus_{k=0}^{\infty} H^0(O(k))$ (with multiplication given by the tensor product and the $\mathbb{Z}_2$-grading induced from $O(k)$) is isomorphic as a $\mathbb{C}$-superalgebra with the free supercommutative superalgebra $\mathbb{C}[z_0 \dots z_{m+n}]$ generated by the homogeneous supercoordinates. This algebra also has a $\mathbb{Z}$-grading given by the degree of monomials in $z_0 \dots z_{m+n}$, and the supervector space $H^0(O(k))$ identifies with the component of degree k with respect to this grading.

A superscalar product $(\, , \,)$ on V makes $\mathbb{P}V$ into a Kähler supermanifold as follows. Since the total space of $O(-1)$ identifies with the affine supermanifold $\mathbb{A}_V$ defined by V, the total space of $O(1)$ identifies with the affine supermanifold $\mathbb{A}_{V^*}$ defined by V^* and thus carries the Hermitian metric induced by the superscalar product $(\, , \,)_*$. The former gives the Hermitian supermetric h on $O(1)$. This determines a Kähler supermetric on $\mathbb{P}V$, known as the *Fubini-Study supermetric* defined by $(\, , \,)$, through the relation:

$$\omega = i\partial\bar{\partial} \ln h(z, z) \ , \tag{2.10}$$

where the natural logarithm $\ln \alpha$ of an element of the Grassmann algebra $\mathbb{C}[z_{m+1}, \dots, z_{m+n}]$ is defined through its power series, when α is invertible in this superalgebra.

When $V = \mathbb{C}^{m+1|n}$ endowed with its canonical superscalar product, then the projective superspace $\mathbb{P}V$ is denoted by $\mathbb{P}^{m|n}$ and called the projective superspace of type $(m|n)$. This is isomorphic as a Hermitian supermanifold to the projective superspace over any supervector space of dimension $(m+1|n)$.

2.4 Hodge supermanifolds and quantum super line bundles

Consider a connected compact complex supermanifold X of dimension $(m|n)$. By definition, a *polarization* of X is a positive holomorphic super line bundle L over X. The following *Kodaira superembedding theorem* was proved in [9]: Given a polarized Kähler supermanifold (X, L), there exists a positive integer k_0 such that the tensor powers $L^k := L^{\otimes k}$ are very ample for all $k \geq k_0$ in the sense that the super Kodaira map defined by any homogeneous basis of the complex supervector space $H^0(L^k)$ is a superembedding of X in the projective superspace $\mathbb{P}[H^0(L^k)^*]$.

As in the case of ordinary manifolds, there is the natural concept of a Hodge supermanifold, providing a connection between Kähler and algebraic supergeometry [6]. A Kähler form ω is called *integral*, if its cohomology class $[\omega]$ belongs to $H^2(X, \mathbb{Z})$. In this case,

[3] A supermanifold $(X, \mathcal{A})$ is called *split* if the sheaf $\mathcal{A}$ is globally isomorphic to $\wedge^n_{\mathcal{A}_{\rm red}} \hat{\mathcal{A}}$ (rather than simply locally isomorphic).

(X,ω) is called a *Hodge supermanifold*. It was shown by Kostant ([6], Prop. 4.10.2) that in this case X admits a positive holomorphic super line bundle L endowed with a connection ∇ such that $\omega = \frac{i}{2\pi}F_\nabla$ where F_∇ is the curvature of ∇ (in particular, we have $[\omega] = c_1(L)$). A triplet (X,ω,L) of this type is called a *polarized Hodge supermanifold*.

Given a polarized Hodge supermanifold (X,L,ω), the super line bundle L carries a Hermitian supermetric h which determines the connection ∇ as its Chern connection, i.e. the unique connection of Dolbeault type $(1,0)$ compatible with h; in fact, h and ∇ essentially determine each other [8]. The quadruple (X,L,h,ω) is called a *prequantized Hodge supermanifold*. Under the replacement $L \to L^k$, we have an induced supermetric $h_k = h^{\otimes k}$ on L^k and the corresponding Chern connection $\nabla_k = \nabla^{\otimes k}$ with curvature $F_{\nabla_k} = \frac{k}{2\pi i}\omega$. Fixing a measure μ on X yields a superscalar product on the vector super space $H^0(L^k)$ [8]:

$$\langle s_1, s_2\rangle_k^{\mu,h} := \int_X d\mu\ h_k(s_1,s_2) \quad . \tag{2.11}$$

The standard choice for $d\mu$ is the super Liouville measure $d\mu_\omega$ defined by the Kähler superform ω. On a super-coordinate chart U with local coordinates $Z^I = z^1,\ldots,z^m,\zeta^1,\ldots,\zeta^n$, we have:

$$d\mu_\omega|_U := (2\pi)^n|\mathrm{sdet}(\omega_{IJ})|dz^1\wedge d\bar z^1\wedge\ldots\wedge dz^m\wedge d\bar z^m id\zeta^1 d\bar\zeta^1\ldots id\zeta^n d\bar\zeta^n\,, \tag{2.12}$$

where ω_{IJ} are the coefficients of the Kähler form $\omega = \omega_{IJ}dZ^I\wedge d\bar Z^J$ and sdet(A) is the superdeterminant (Berezinian) of the supermatrix A. However, it is is desirable to work in a more general setting in order to include e.g. the case when X is algebraically a Calabi-Yau supermanifold and μ_{CY} is the volume form determined by its the holomorphic volume element.

Remark. As for ordinary manifolds, we can derive a local Kähler potential K from the Hermitian bundle supermetric h. Given a global supersection σ of L, we let $K_\sigma := -\log h(\sigma,\sigma)$, which is a Kähler potential on the set $U_\sigma := \{x\in X|\sigma(x)\in \underline{\mathcal{O}}_x^\times\}$ (here $\underline{\mathcal{O}}_x^\times$ is the supercommutative group of invertible elements in the local superalgebra $\underline{\mathcal{O}}_x$):

$$\omega = \frac{i}{2\pi}\partial\bar\partial K_\sigma = \frac{1}{2\pi i}\log h(\sigma,\sigma) = \frac{i}{2\pi}F_\nabla\ . \tag{2.13}$$

2.5 Parameterizing hermitian bundle supermetrics and polarized Kähler super forms

Let us fix a polarized complex supermanifold (X,L), where $L\stackrel{\pi}{\to}X$ is a positive holomorphic super line bundle on X. Since L has rank $(1,0)$, we have that $L_{\mathrm{res}} = L_{\mathrm{red}} \stackrel{\pi_{\mathrm{red}}}{\to} X_{\mathrm{red}}$ is an ordinary holomorphic line bundle on the complex manifold X_{red}. We let $L_{\mathrm{red}}^\times \stackrel{\pi_{\mathrm{red}}^\times}{\to} X_{\mathrm{red}}$ be the total space L_{red} with the zero supersection removed. A Hermitian supermetric h on L is determined by the associated Hermitian $\mathbb{C}$-sesquilinear maps $h_x : (L_{\mathrm{red}})_x\times(L_{\mathrm{red}})_x\to\underline{\mathcal{C}}_x$ on the fibers of L_{red} at the points x of X_{red}. Hence h is uniquely determined by the global supersection $\hat h$ of $(\pi_{\mathrm{red}}^\times)^*(\underline{\mathcal{C}})$ given by:

$$\hat h(q) := h(q,q)\in\mathcal{C}_{\pi_{\mathrm{red}}^\times(q)}\ \ ,\quad q\in L_{\mathrm{red}}^\times\quad . \tag{2.14}$$

where $\underline{\mathcal{C}}$ is the sheaf of smooth superfunctions on X, viewed as a sheaf of $\mathcal{C}$-superalgebras on $X_{\rm red}$. Notice that the right hand side is invertible in the superalgebra $\underline{\mathcal{C}}_{\pi_{\rm red}(q)}$. We have $\hat{h}(cq) = |c|^2\hat{h}(q)$ for all $q \in L^\times_{\rm red}$ and all $c \in \mathbb{C}^*$. The set $\mathrm{Met}(L)$ of Hermitian supermetrics on L can be identified with the set of all such $\hat{h}$. If we fix a reference super metric h_0 on L, then any other supermetric h is described by the global supersection $\phi = \frac{\hat{h}}{\hat{h}_0}$ of $\underline{\mathcal{C}}$, which is a smooth superfunction on X whose body (projection to $\mathcal{C}(X)$) is a positive-definite ordinary smooth function. We thus find that $\mathrm{Met}(L)$ can be identified with the real cone $\underline{\mathcal{C}}_{>0}(X)$ of all such smooth superfunctions.

In this paper, we will use a slightly different parameterization when L is very ample. For any $q \in L^\times_{\rm red}$, we let $\hat{q} : H^0(L) \to \underline{\mathcal{O}}_{\pi(q)}$ be the $\mathbb{C}$-linear functional (called *evaluation functional*) defined through:

$$s(\pi(q)) = \hat{q}(s)q\,, \quad s \in H^0(L)\ . \tag{2.15}$$

We have the obvious property $\widehat{cq} = \frac{1}{c}\hat{q}$ for all non-vanishing complex numbers c. The very ampleness of L implies $\hat{q} \neq 0$ for all $q \in L^\times_{\rm red}$.

Consider a Hermitian superscalar product $(\ ,\ )$ on $E := H^0(L)$, where $\dim_{\mathbb{C}}(E) = (m+1|n)$. Picking a homogeneous basis $s_0, \ldots, s_{m+n}$ of E with $s_0 \ldots s_m$ even and $s_{m+1} \ldots s_{m+n}$ odd, we let $G_{ij} = (s_i, s_j) \in \mathbb{C}$. Also let G^{ij} be the inverse of G_{ij}, so that $\sum_{j=0}^{m+n} G^{ij}G_{jk} = \delta^i_k$. Since the superscalar product is even, the matrix G is block-diagonal:

$$G_{i\iota} = G_{\iota i} = 0 \ \text{ for } \ i = 0\ldots m, \ \ \iota = m+1\ldots m+n \ \ .$$

The super-Hermitian property of $(\ ,\ )$ reads:

$$\begin{aligned} \overline{G}_{ij} &= G_{ji} \qquad i,j = 0\ldots m \\ \overline{G}_{\iota\gamma} &= -G_{\gamma\iota} \quad \iota,\gamma = m+1\ldots m+n \ \ . \end{aligned}$$

Furthermore, the submatrix $(G_{ij})_{i,j=0\ldots m}$ is positive-definite.

Let us fix a point $q \in L^\times_{\rm red}$ with $\pi(q) = x$. As explained in section 2.1., the pairing $(\ ,\ )$ induces a superscalar product $(\ ,\ )_*$ on the dual space $H^0(L)^* = \mathrm{Hom}_{\mathbb{C}}(H^0(L), \mathbb{C})$. The latter extends uniquely to a Hermitian bilinear form $(\!(\ ,\ )\!)_x$ on the $\underline{\mathcal{O}}_x$-module $\mathrm{Hom}_{\mathbb{C}}(H^0(L), \underline{\mathcal{O}}_x) = H^0(L)^* \otimes_{\mathbb{C}} \underline{\mathcal{O}}_x$. This allows us to consider the Hermitian supermetric h_B on L whose 'square norm' superfunction is given by (this is well-defined since the denominator is invertible in the local algebra $\mathcal{O}_x$):

$$\hat{h}_B(q) = (\!(\hat{q}, \hat{q})\!)_x = \frac{1}{\sum_{i,j=0}^{m+n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} \in \underline{\mathcal{O}}_x \quad , \quad q \in L^\times_{\rm red},\ x = \pi(q) \ \ . \tag{2.16}$$

This is called the *Bergman supermetric* on L defined by $(\ ,\ )$. Since we now have a reference Hermitian supermetric on L, we can describe any other supermetric h via the superfunction:

$$\epsilon := \frac{\hat{h}}{\hat{h}_B} \in \underline{\mathcal{C}}_{>0}(X)\ , \tag{2.17}$$

which we call the *epsilon superfunction of h relative to* $(\ ,\ )$:

$$h(q,q) = \epsilon(\pi(q))h_B(q,q) \ \ . \tag{2.18}$$

More explicitly, we have $\epsilon(x) = \sum_{i,j=0}^{m+n} G^{ij} h(x)(s_i(x), s_j(x))$. Thus, Hermitian supermetrics on L are parameterized by their relative epsilon superfunctions, once one fixes a superscalar product on $H^0(L)$.

The relative epsilon superfunction defined above depends on h and on the superscalar product chosen on $H^0(L)$ and is a generalization of the more familiar object considered in [21–23]. To make contact with the latter, notice that fixing h gives a distinguished choice of a superscalar product on $H^0(L)$, namely the L^2 superscalar product $\langle\, ,\, \rangle$ defined by h and by the Liouville density of the associated super Kähler form ω. The epsilon superfunction of h with respect to this superscalar product depends only on h (remember that ω is determined by h), and will be called the *absolute epsilon superfunction* of h. The latter generalizes the absolute epsilon function considered in [21–23].

The L-polarized Kähler supermetric on X determined by h_B is called the *Bergman supermetric on X induced by* $(\, ,\,)$. Its Kähler super form is denoted by ω_B. The Kähler super form ω determined by the Hermitian bundle supermetric (2.18) takes the form:

$$\omega = \omega_B - \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon \ ,$$

so as expected we have $\omega = \omega_B$ iff the relative epsilon superfunction of h is constant. Since ω determines h up to multiplication by a constant, it also determines the relative epsilon superfunction of the latter up to the same ambiguity. We shall see below that L-polarized Bergman supermetrics are those supermetrics induced on X by pulling-back Fubini-Study supermetrics through the Kodaira superembedding $i : X \hookrightarrow \mathbb{P}^{m|n}$ (where $\dim_{\mathbb{C}}(H^0(L)) = (m+1|n)$) determined by the very ample super line bundle L, where the Fubini-Study supermetric being pulled-back is determined by the superscalar product on $H^0(L)^*$.

Remark. The Hermitian superscalar product on $H^0(L)$ defined by h_B and by the volume form of ω_B:

$$\langle s,t\rangle = \int_X d\mu_{\omega_B} h_B(s,t) \quad (s,t \in H^0(L))$$

need not coincide with the superscalar product $(\, ,\,)$ which parameterizes h_B. If they do, then one says that the superscalar product $(\, ,\,)$ and associated Bergman bundle and manifold supermetrics h_B, ω_B are *balanced* [24, 25]. Clearly, for ω_B to be balanced, the *absolute* epsilon superfunction has to be constant.

2.6 Bergman supermetrics from metrized Kodaira superembeddings

Let X be a compact complex supermanifold. By the Kodaira superembedding theorem, a positive super line bundle L on X gives a holomorphic superembedding $i : X \hookrightarrow \mathbb{P}V$, where $E := H^0(L)$ and $V = E^*$ is the space of holomorphic supersections of L, whose complex superdimension we denote by $(m+1|n)$. The embedding allows us to view X as a nonsingular projective supervariety in $\mathbb{P}V$, whose homogeneous coordinate ring $R(X,L) = \oplus_{k\geq 0} H^0(L^k)$ is generated in monomial degree $k = 1$. In particular, L and the pull-back $i^*(H)$ of the hyperplane superbundle $H := O_{\mathbb{P}V}(1)$ are isomorphic as holomorphic super line bundles.

Conversely, if we are given any smooth projective supervariety X in a projective superspace $\mathbb{P}V$ whose vanishing ideal $I(X)$ is generated in monomial degrees greater than one, then the restriction $O_X(1) = O_{\mathbb{P}V}(1)|_X$ is very ample and the embedding $X \hookrightarrow \mathbb{P}V$ can be viewed as the Kodaira superembedding determined by this restriction. The space of holomorphic supersections of $O_X(1)$ identifies with the supervector space $E = V^*$.

A *metrized Kodaira embedding* is a Kodaira superembedding determined by a very ample super line bundle L on X together a fixed choice of a Hermitian superscalar product $(\ ,\ )$ on its space of holomorphic supersections $E := H^0(L)$. For such embeddings, the superscalar product on E induces a superscalar product on $V = E^*$, which makes $\mathbb{P}V$ into a (finite-dimensional) projective super Hermitian space. The later carries the Fubini-Study supermetric[4] determined by the superscalar product. Its Kähler super form is given by:

$$\pi^*(\omega_{\mathrm{FS}})(v) = \frac{i}{2\pi}\partial\bar{\partial}\log[(v,v)_*] \ ,$$

where $\pi : V \to \mathbb{P}V$ is the canonical projection while $(\ ,\ )_*$ is the superscalar product induced on $V = E^*$. There exists a one to one correspondence between metrized Kodaira superembeddings of X and holomorphic superembeddings in finite-dimensional projective super Hermitian spaces such that the vanishing ideal of the superembedding is generated in monomial degrees greater than one.

The Fubini-Study supermetric admits the hyperplane superbundle H as a quantum line superbundle, when the latter is endowed with the Hermitian bundle supermetric h_{FS} induced from E. Since $L \simeq i^*(H)$ as holomorphic super line bundles, the pull-back $i^*(h_{\mathrm{FS}})$ defines a Hermitian supermetric h_B on L. The latter coincides with the Bergman bundle supermetric determined by $(\ ,\ )$. The pulled-back Kähler form $\omega_B = i^*(\omega_{\mathrm{FS}})$ admits (L, h_B) as a quantum super line bundle, and coincides with the Bergman Kähler form determined by $(\ ,\ )$. It follows that Bergman supermetrics on X coincide with pull-backs of Fubini-Study supermetrics via metrized Kodaira embeddings.

Remark. A choice of homogeneous basis $z_0 \ldots z_{m+n}$ for $E = V^*$ allows us to express $v \in V$ as: $v = \sum_{i=0}^{m+n} v_i e_i$, where (e_i) is the homogeneous basis of V dual to (z_i) and $v_i = z_i(v)$. This gives an identification of V with the supervector space $\mathbb{C}^{m+1|n}$ endowed with the superscalar product given by $\langle u, v\rangle = \sum_{i,j=0}^{m+n} G^{ij}\bar{u}_i v_j$, where the G^{ij} are given as above. Then $\mathbb{P}V$ identifies with $\mathbb{P}^{m+1|n}$ endowed with the Fubini-Study supermetric defined by this superscalar product. It is customary to choose an orthonormal basis, in which case the Fubini-Study supermetric takes the familiar form in homogeneous supercoordinates. In this case, the freedom of choosing the superscalar product $(\ ,\ )$ is replaced by the freedom of acting with $\mathrm{PGL}_{\mathbb{C}}(m+1|n)$ transformations on the homogeneous supercoordinates of $\mathbb{P}^{m|n}$.

3 Generalized Berezin and Toeplitz quantization of supermanifolds

In this section, we extend the generalized Berezin and Toeplitz quantization procedure of [10] to the case of Kähler supermanifolds. The subtle point of this extension is to

[4] Homogeneous Kähler supermetrics on $\mathbb{P}V$ are in bijection with super Hermitian scalar products on E taken up to constant rescaling, and these are the Fubini-Study supermetrics. They are all related by $\mathrm{PGL}(E)$-transformations.

control the various super Hermitian forms involved in defining the Rawnsley supercoherent projectors.

In the following, let X be a Kähler supermanifold endowed with a fixed very ample super line bundle L. We also fix a Hermitian superscalar product on the finite-dimensional supervector space $E = H^0(L)$, whose dimension we denote by $(m+1|n)$.

We will consider a homogeneous basis $s_0 \ldots s_m, s_{m+1} \ldots s_{m+n}$ of E, where $s_0 \ldots s_m$ are even and $s_{m+1} \ldots s_{m+n}$ are odd. We let G be the Hermitian matrix with entries $G_{ij} := (s_i, s_j)$ and G^{ij} be the entries of the inverse matrix G^{-1}. Any supersection $s \in E$ can be expanded as:

$$s = \sum_{i,j=0}^{m+n} G^{ij}(s_j, s)s_i \quad .$$

As mentioned above, the matrix G is block-diagonal because the bilinear form $(\ ,\ )$ is even, and the submatrix $(G_{ij})_{i,j=0\ldots m}$ is positive-definite.

3.1 Supercoherent states

Recall that $\underline{\mathcal{O}}(L)$ denotes the sheaf of holomorphic supersections of L. For any point $x \in X$, the stalk $\underline{\mathcal{O}}_x(L)$ of this sheaf at x is a free $\underline{\mathcal{O}}_x$-supermodule of rank $(1|0)$. We let $\underline{\mathcal{O}}_x^{\times,ev}(L)$ be the set of even bases of this module, i.e. the set of even elements $q \in L_x$ such that $\underline{\mathcal{O}}_x(L) = \underline{\mathcal{O}}_x q$. We have $\underline{\mathcal{O}}_x^{\times,ev}(L) = \underline{\mathcal{O}}_x^{\times,ev} q$ for any $q \in (L_{\mathrm{red}}^\times)_x$, where $\underline{\mathcal{O}}_x^{\times,ev}$ is the set of *even* invertible elements of $\underline{\mathcal{O}}_x$ (this is a subgroup of the multiplicative monoid of the superalgebra $\underline{\mathcal{O}}_x$).

Given $q \in (\mathbb{L}_{\mathrm{red}}^\times)_x$ and a supersection $s \in E = H^0(L)$, we have:

$$s(x) = \hat{q}(s)q \tag{3.1}$$

for some element $\hat{q}(s) \in \underline{\mathcal{O}}_x$. This gives an even $\mathbb{C}$-linear functional $\hat{q} : E \to \underline{\mathcal{O}}_x$. Consider the $\underline{\mathcal{O}}_x$-supermodule $E_x := \underline{\mathcal{O}}_x \otimes_{\mathbb{C}} E$. The superscalar product on E extends to a nondegenerate and even $\underline{\mathcal{O}}_x$-sesquilinear map $\langle\!\langle\ ,\ \rangle\!\rangle_x : E_x \times E_x \to \underline{\mathcal{C}}_x$ as follows:

$$\langle\!\langle \alpha \otimes s, \beta \otimes t \rangle\!\rangle_x = (-1)^{\tilde{s}\tilde{\beta}}(\bar{\alpha}\beta) \otimes_{\mathbb{C}} (s,t) \quad . \tag{3.2}$$

where $\alpha, \beta \in \underline{\mathcal{O}}_x$ and $s, t \in E$. These extended even pairings make each E_x into a Hermitian $\underline{\mathcal{O}}_x$-module.

By the Riesz theorem, we have a uniquely determined element $e_q \in E_x$ such that:

$$\langle\!\langle e_q, s \rangle\!\rangle_x = \hat{q}(s) \quad \forall s \in E \quad , \tag{3.3}$$

where we consider $s \in E$ tensored with the identity $1_{\underline{\mathcal{O}}_x}$ in E_x. Direct computation gives the explicit expression:

$$e_q = \sum_{i,j=0}^{m+n} G^{ji}\overline{\hat{q}(s_i)} \otimes_{\mathbb{C}} s_j \quad ,$$

which implies:

$$\langle\!\langle e_q, e_q \rangle\!\rangle_x = \sum_{i,j=0}^{m+n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j) \in \underline{\mathcal{O}}_x \quad .$$

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Notice that e_q cannot be the zero supersection, since that would imply that all supersections of L (and thus of $L_{\rm red}$) vanish at x, which is impossible since $L_{\rm red}$ is very ample. Also notice that $\langle\!\langle e_q, e_q\rangle\!\rangle_x$ belongs to $(\underline{\mathcal{C}}_{>0})_x \subset \underline{\mathcal{C}}_x^\times$. Indeed, the fact that $\hat{q}$ is an even map implies that e_q itself is even and we have:

$$\mathrm{ev}_x(\langle\!\langle e_q, e_q\rangle\!\rangle_x) = \sum_{i,j=0}^{m} G^{ij}\overline{\mathrm{ev}_x(\hat{q}(s_i))}\mathrm{ev}_x(\hat{q}(s_j)) \ , \tag{3.4}$$

which is positive since the restriction of $(\ ,\ )$ to E_+ is positive-definite and because $L_{\rm red}$ is very ample. The element e_q of E_x will be called the *Rawnsley supercoherent vector* defined by q. This generalizes the coherent vectors introduced in [21] to the supermanifold case.

If q' is another element of $(L^\times_{\rm red})_x$, then $q' = cq$ for some $c \in \mathbb{C}$ and we have $e_{q'} = \frac{1}{\bar{c}} e_q$. It follows that the rank $(1|0)$ $\underline{\mathcal{O}}_x$-module $l_x := \langle e_q\rangle = \underline{\mathcal{O}}_x e_q \subset E_x$ depends only on the point $x \in X$. This can be interpreted as follows. Let $\bar{L}$ be the super line bundle obtained by reversing the complex structure of all fibers; this is a holomorphic super line bundle over the complex supermanifold $\bar{X}$ obtained by reversing the complex structure of X. The scaling property of supercoherent vectors implies that the element $e_x := \bar{q} \otimes_{\mathbb{C}} e_q \in \bar{L}_x \otimes_{\mathbb{C}} E$ depends only on the point $x \in X$. The superscalar product on E extends to a sesquilinear map $\langle\!\langle\ ,\ \rangle\!\rangle$ taking $[\bar{L}_x \otimes_{\mathbb{C}} E] \times [\bar{L}_y \otimes_{\mathbb{C}} E]$ into $\bar{L}_x \otimes_{\mathbb{C}} \bar{L}_y$. So in particular, the combination $K(x, y) = \langle\!\langle e_x, e_y\rangle\!\rangle$ defines a holomorphic supersection K of the external tensor product $\bar{L} \boxtimes \bar{L}$ (which is a holomorphic super line bundle over the supermanifold $\bar{X} \times \bar{X}$). This will be called the *reproducing kernel* of the finite-dimensional super Hermitian space $(E, (\ ,\ ))$.

Rawnsley's *supercoherent projectors* are the $\underline{\mathcal{O}}_x$-linear 'orthoprojectors' $P_x \in \mathrm{End}_{\underline{\mathcal{O}}_x}(E_x)$ on the rank one submodules $l_x \subset E_x$:

$$P_x(\mathbf{s}) = \frac{e_q\langle\!\langle e_q, s\rangle\!\rangle_x}{\langle\!\langle e_q, e_q\rangle\!\rangle_x} \in l_x \quad (s \in E_x) \tag{3.5}$$

These are well-defined since $\langle\!\langle e_q, e_q\rangle\!\rangle_x$ is an even invertible element of $\underline{\mathcal{C}}_x$. The projectors depend only on L, on the point $x \in X$ and on the superscalar product chosen on E. Given a $\mathbb{C}$-linear operator $C \in \mathrm{End}(E)$, its *lower Berezin symbol* is the smooth superfunction $\sigma(C) \in \underline{\mathcal{C}}(X)$ given by:

$$\sigma(C)(x) := \mathrm{str}(CP_x) = \frac{\langle\!\langle e_q, Ce_q\rangle\!\rangle_x}{\langle\!\langle e_q, e_q\rangle\!\rangle_x} \ . \tag{3.6}$$

This gives a $\mathbb{C}$-linear map $\sigma : \mathrm{End}(E) \to \underline{\mathcal{C}}(X)$, whose image we denote by Σ. Notice that σ and Σ depend only on L and on the superscalar product $(\ ,\ )$ chosen on E. The obvious property:

$$\sigma(C^\dagger) = \overline{\sigma(C)}$$

implies that Σ is closed under the complex conjugation of the superalgebra $\underline{\mathcal{C}}(X)$, i.e. we have $\bar{\Sigma} = \Sigma$. Also notice that Σ contains the constant unit function $1_X = \sigma(\mathrm{id}_E)$ and that σ is an even map:

$$\widetilde{\sigma(C)} = \tilde{C} \ , \tag{3.7}$$

when C is a $\mathbb{Z}_2$-homogeneous operator in E. It follows that Σ is a $\mathbb{Z}_2$-homogeneous subspace of $\underline{\mathcal{C}}(X)$, i.e. $\Sigma = \Sigma_+ \oplus \Sigma_-$ with $\Sigma_\pm = \Sigma \cap \underline{\mathcal{C}}(X)_\pm$.

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3.2 Generalized Berezin quantization

As for ordinary manifolds, the Berezin symbol map $\sigma : \mathrm{End}(E) \to \underline{\mathcal{C}}(X)$ is injective so its kernel is trivial. This is easily seen from expanding

$$\langle\!\langle e_q, Ce_q \rangle\!\rangle_x = \sum_{i,j,k,l} (G^{ji}\hat{q}(s_i))^* G^{kl}\hat{q}(s_l)(s_j, Cs_k) \ . \tag{3.8}$$

As P_x is independent of the choice of q for every x, we choose q to be $(1, x)$ everywhere. Also note that we can write C in the above equation as $C = \sum_{i,j} |s_i) C^{ij} (s_j|$. Altogether we thus obtain

$$\langle\!\langle e_q, Ce_q \rangle\!\rangle_x = \sum_{i,j} C^{ij} \bar{s}_i(x) s_j(x) \ . \tag{3.9}$$

As the (s_i) form a (holomorphic) basis of E, it follows that if (3.9) equals zero then $C^{ij} = 0$.

It follows that the corestriction $\sigma|^{\Sigma} : \mathrm{End}(E) \to \Sigma$ is an isomorphism of supervector spaces and we can associate an operator on E to every superfunction $f \in \Sigma$ via the *Berezin quantization map* $Q = (\sigma|^{\Sigma})^{-1} : \Sigma \to \mathrm{End}(E)$:

$$Q(f) := \sigma^{-1}(f) \quad \forall f \in \Sigma \ . \tag{3.10}$$

The quantization map Q is even and depends only on L and on the choice of superscalar product on $H^0(L)$. It satisfies the relations:

$$Q(\bar{f}) = Q(f)^\dagger \ , \quad Q(1_X) = \mathrm{id}_E \ .$$

The Berezin superalgebra. The *Berezin product* $\diamond : \Sigma \times \Sigma \to \Sigma$ is defined via the formula:

$$f \diamond g := \sigma(Q(f)Q(g)) \Leftrightarrow Q(f \diamond g) = Q(f)Q(g) \ . \tag{3.11}$$

Together with the complex conjugation of smooth superfunctions $f \to \bar{f}$, it makes Σ into a unital finite-dimensional associative $*$-superalgebra (in particular, the conjugation of $\underline{\mathcal{C}}(X)$ restricts to an even involution of Σ). The Berezin quantization map gives an isomorphism of $*$-superalgebras:

$$Q : (\Sigma, \diamond, {}^-) \to (\mathrm{End}(E), \circ, \dagger) \ .$$

Recall that $(\mathrm{End}(E), \circ, \dagger)$ is a $*$-superalgebra with nondegenerate trace given by the usual supertrace. It follows that the induced linear map (called the *Berezin supertrace*):

$$\int f := \mathrm{str}\ Q(f) \quad (f \in \Sigma) \tag{3.12}$$

is a nondegenerate supertrace on the Berezin superalgebra $(\Sigma, \diamond, {}^-)$:

$$\begin{aligned} \int \bar{f} &= \overline{\int f} \ , \\ \int f \diamond g &= (-1)^{\tilde{f}\tilde{g}} \int g \diamond f \ , \\ \int f \diamond g &= 0, \ \forall g \in \Sigma \Rightarrow f = 0 \ . \end{aligned}$$

The super Hermitian pairing on Σ (called the *Berezin pairing*) obtained by transporting the super Hilbert-Schmidt pairing:

$$\prec f, g \succ_B := \langle Q(f), Q(g)\rangle_{\mathrm{HS}} = \mathrm{str}\left(Q(f)^\dagger Q(g)\right) \tag{3.13}$$

coincides with the pairing induced by the Berezin supertrace:

$$\prec f, g \succ_B = \int \bar{f} \diamond g \quad .$$

Notice that $\prec 1_X, 1_X \succ_B = \langle \mathrm{id}_E, \mathrm{id}_E\rangle_{\mathrm{HS}} = m - n + 1$, where $(m+1|n)$ was the superdimension of E.

The squared two point superfunction. For later reference, we define the superanalogue of the squared two-point function of coherent states. For this, note that the $\underline{\mathcal{O}}_x$-sesquilinear maps $\langle\!\langle\, ,\, \rangle\!\rangle_x$ on E_x (see equation (3.2)), uniquely extend further to a pairing $\langle\!\langle\, ,\, \rangle\!\rangle_{x,y} : [\underline{\mathcal{O}}_x \otimes_{\mathbb{C}} E] \times [\underline{\mathcal{O}}_y \otimes_{\mathbb{C}} E] \to \bar{\mathcal{O}}_x \otimes \mathcal{O}_y$. Define the two-point superfunction $\Psi : X \times X \mapsto \underline{\mathcal{C}}_x \otimes \underline{\mathcal{C}}_y$ via:

$$\Psi(x, y) := \mathrm{str}(P_x P_y) = \sigma(P_x)(y) = \sigma(P_y)(x) = \frac{\langle\!\langle e_x, e_y\rangle\!\rangle_{x,y}\overline{\langle\!\langle e_x, e_y\rangle\!\rangle}_{x,y}}{\langle\!\langle e_x, e_x\rangle\!\rangle_x \langle\!\langle e_y, e_y\rangle\!\rangle_y} \quad . \tag{3.14}$$

As the supercoherent state projectors P_x, P_y are even operators on E, Ψ is symmetric on $X \times X$:

$$\Psi(x, y) = \Psi(y, x) \quad \forall x, y \in X$$

and vanishes at points (x, y) where the directions of the supercoherent vectors e_x and e_y in E are orthogonal to each other with respect to the pairing $\langle\!\langle\, ,\, \rangle\!\rangle_{x,y}$.

3.3 Changing the superscalar product in generalized Berezin quantization

Any Hermitian superscalar product $(\, ,\,)'$ on E has the form:

$$(s, t)' = (As, t) \tag{3.15}$$

with A a $(\, ,\,)$-super Hermitian even invertible operator. Such an operator has the block diagonal structure $A = A_+ \oplus A_-$ (with $A_\pm \in GL(E_\pm)$) where $A_\pm$ are $(\, ,\,)_\pm$-Hermitian and A_+ is positive definite. The supercoherent states with respect to the new product $(\, ,\,)'$, which in turn induces the pairing $\langle\!\langle\, ,\, \rangle\!\rangle'_x$, are given by:

$$e'_q = A^{-1} e_q \quad (q \in L_x^\times) \quad , \tag{3.16}$$

while the new supercoherent projectors take the form:

$$P'_x = \frac{1}{\sigma(A^{-1})(x)} A^{-1} P_x \quad (x \in X) \quad . \tag{3.17}$$

The symbol $\sigma(A^{-1})(x) = \frac{(e_q|A^{-1}|e_q)_x}{(e_q|e_q)_x}$ of A^{-1} computed with respect to $\langle\!\langle\, ,\, \rangle\!\rangle_x$ and the symbol $\sigma'(A)(x) = \frac{(e'_q|A|e'_q)'_x}{(e'_q|e'_q)'_x}$ of A computed with respect to $\langle\!\langle\, ,\, \rangle\!\rangle'_x$ are related by:

$$\sigma(A^{-1})(x) = \frac{1}{\sigma'(A)(x)} \quad . \tag{3.18}$$

As A_+ is positive definite, the body of both $\sigma(A)$ and $\sigma'(A)$ is non-vanishing. Therefore, both these superfunctions are invertible on X. Given an operator C, we have more generally:

$$\sigma'(C) = \frac{\sigma(CA^{-1})}{\sigma(A^{-1})} \tag{3.19}$$

and:

$$\sigma(C) = \frac{\sigma'(CA)}{\sigma'(A)} \quad . \tag{3.20}$$

Since A is even, so are $\sigma'(A)$ and $\sigma(A^{-1})$. Therefore, the order of multiplication in the fractions is irrelevant. Let Q' be the Berezin quantization map defined by $(\ ,\ )'$ and $\Sigma' \subset \underline{\mathcal{C}}(X)$ be the image of σ'. Equation (3.19) shows that

$$\Sigma' = \frac{1}{\sigma(A^{-1})} \cdot \Sigma = \left\{ \frac{1}{\sigma(A^{-1})} \cdot f \mid f \in \Sigma \right\}$$

and that:

$$Q'(f) = Q(\sigma(A^{-1})f)A \quad \forall f \in \Sigma' \quad .$$

As for ordinary manifolds, we have the following proposition, whose proof mimics that of the corresponding result of [10]:

Proposition. The Berezin quantizations defined by two different superscalar products on E agree iff A is proportional to the identity, i.e. iff the two superscalar products are related by a constant scale factor $\lambda \in \mathbb{C}^*$. In this case, the supercoherent states differ by a constant homothety and the Rawnsley supercoherent projectors are equal.

3.4 Integral representations of the superscalar product

In the following, let L be a very ample super line bundle, $E = H^0(L)$ the vector space of supersections and $(\ ,\ )$ a superscalar product on E. We will look for those superscalar products $(\ ,\ )$ which admit integral representations through a measure μ and a Hermitian supermetric h on L; such a representation is required for defining generalized Toeplitz quantization.

A Hermitian bundle supermetric h on L can be parameterized by its epsilon superfunction relative to $(\ ,\ )$:

$$\epsilon(x) := h(x)(q,q)\ (\!(e_q, e_q)\!)_x = \sum_{i,j=0}^{m+n} G^{ij} h(x)(s_i(x), s_j(x)) \quad . \tag{3.21}$$

Note that the right hand side is indeed independent of the choice of q. Furthermore, $h(x)$ is uniquely determined by $h(q,q)$; conversely, the epsilon superfunction determines a Hermitian supermetric via $h(x)(q,q) = \frac{\epsilon(x)}{(\!(e_q, e_q)\!)_x}$.

Let us look for integral representations of (s,t) of the following form:

$$(s,t) = \int_X d\mu(x)\ h(x)(s(x), t(x)) \quad .$$

Since the right hand side equals $\int_X d\mu(x)\epsilon(x)(\!(s, P_x t)\!)_x$ we have:

Proposition. The superscalar product $(\, , \,)$ on E coincides with the L^2 product induced by (μ, h) iff the relative epsilon superfunction of the pair $(h, (\, , \,))$ satisfies the identity

$$\int_X d\mu(x)\epsilon(x)P_x = \mathrm{id}_E \quad , \tag{3.22}$$

i.e. iff the supercoherent states defined by $(\, , \,)$ form an 'overcomplete set' with respect to the measure $\mu_\epsilon = \mu\epsilon$.

The precise mathematical meaning of equation (3.22) is as follows. Recall that the supercoherent projectors P_x are $\underline{\mathcal{O}}_x$-linear operators acting in the free modules $E_x = \underline{\mathcal{O}}_x \otimes_{\mathbb{C}} E$, i.e. elements of the free $\underline{\mathcal{O}}_x$-module $\mathrm{End}_{\underline{\mathcal{O}}_x}(E_x) \cong \underline{\mathcal{O}}_x \otimes_{\mathbb{C}} \mathrm{End}(E)$. The map P which associates the operator P_x to every point of x ($P(x) := P_x$) is a holomorphic supersection of the trivial bundle $X \times \mathrm{End}(E)$, i.e. an element of the free $\underline{\mathcal{O}}(X)$-module $\underline{\mathcal{O}}(X) \otimes_{\mathbb{C}} \mathrm{End}(E)$, while its product with the epsilon superfunction is a smooth supersection of the same bundle and thus an element of the free $\underline{\mathcal{C}}(X)$-module $\underline{\mathcal{C}}(X) \otimes_{\mathbb{C}} \mathrm{End}(E)$. On the other hand, integration of superfunctions over X with respect to μ gives an even $\mathbb{C}$-linear map:

$$\int d\mu : \underline{\mathcal{C}}(X) \to \mathbb{C} \quad ,$$

which extends uniquely to an even $\mathrm{End}(E)$-linear map

$$\left(\int d\mu\right) \otimes_{\mathbb{C}} \mathrm{id}_E : \underline{\mathcal{C}}(X) \otimes_{\mathbb{C}} \mathrm{End}(E) \to \mathrm{End}(E) \quad .$$

In equation (3.22) as well as below, this latter map is denoted simply by $\int d\mu$. It follows that condition (3.22) can be viewed as a spectral decomposition equation for the identity operator of E with a superfunction-valued spectral measure, i.e. a spectral decomposition taken over the $\underline{\mathcal{C}}(X)$-module $\underline{\mathcal{C}}(X) \otimes_{\mathbb{C}} \mathrm{End}(E)$.

Since the Berezin symbol map is injective, condition (3.22) is equivalent to the following (super) Fredholm equation of the first kind:

$$\int_X d\mu(y)\Psi(x, y)\epsilon(y) = 1 \qquad (x \in X) \quad ,$$

where $\Psi(x, y)$ is the squared two point superfunction (3.14).

When the superscalar product on E is fixed, equation (3.22) can be viewed as a constraint on those pairs (μ, h) which allow for an integral representation of this product. Taking the supertrace, we find a normalization condition on the epsilon superfunction:

$$\int_X d\mu(x) \; \epsilon(x) = m + 1 - n \quad ,$$

where the dimension of E is $(m + 1|n)$. More details on this formula for the case when ϵ is constant, in particular when case $m + 1 - n = 0$, are found in section 4.4. Equation (3.22) also allows us to establish a precise relationship between the supertrace on $\mathrm{End}(E)$ and the integral over X:

$$\mathrm{str}(C) = \int_X d\mu(x) \; \epsilon(x)\sigma(C)(x) \quad . \tag{3.23}$$

Here σ is the Berezin symbol map defined by the superscalar product $(\,,\,)$.

Choosing a basis s_i, $i = 0, m+n$, of E, we can rewrite condition (3.22) as a system of inhomogeneous linear integral equations for ϵ:

$$\int_X d\mu\ \epsilon(x) \frac{\overline{\hat{q}(s_i)}\hat{q}(s_j)}{\sum_{i,j=0}^{m+n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} = G_{ij} \quad .$$

These equations, of which a subset are independent, admit an infinity of solutions for the epsilon superfunction, so there is an infinity of Hermitian supermetrics h on L which allow us to represent a given superscalar product $(\,,\,)$ as an integral with respect to μ_ϵ. Note that any such integral representation allows one to extend the superscalar product $(\,,\,)$ to a Hermitian (but possibly degenerate) pairing on the space $\Gamma(L)$ of smooth global supersections of L.

The relative balance condition. The notion of balanced metric (see [24]) can be extended to the case of supermanifolds, as done e.g. in [25]: We say that a superscalar product on E is *μ-balanced* if equation (3.22) admits a constant solution $\epsilon = \frac{m+1-n}{\mu(X)}$, i.e. if the following condition is satisfied:

$$\int_X d\mu(x) P_x \sim \mathrm{id}_E \quad \Leftrightarrow \quad \int_X d\mu\ \frac{\overline{\hat{q}(s_i)}\hat{q}(s_j)}{\sum_{i,j=0}^{m+n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} \sim G_{ij}\ .$$

Fixing the proportionality constant can be sometimes subtle, because for some choices of measure μ one can have $\mu(X) = 0$. For example, the latter phenomenon occurs for some Hodge supermanifolds with respect to the super-Liouville measure determined by their Kähler form, see e.g. the discussion of the normalization of the Liouville measure in section 4.4.

Let ω_h be the L-polarized Kähler form on X determined by a Hermitian superscalar product h on L, and let $\mu_h := \mu_{\omega_h}$ be the Liouville measure on X defined by ω_h. We say that $(\,,\,)$ is *balanced* if it is μ_h-balanced. This is the case considered in [24, 26] as mentioned in section 2.4.

Remarks. It should be stressed that, contrary to the case of ordinary Rawnsley supercoherent states, the supercoherent states e_q do not form an overcomplete basis for E in the classical sense, because the spectral decomposition in that equation is not over E but over the auxiliary module $\underline{\mathcal{C}}(X) \otimes_{\mathbb{C}} \mathrm{End}(E)$. In fact, P_x do not even act on the space E, but on the associated supermodules E_x !

Note also that we have considered a number of different Hermitian pairings on the space $\underline{\mathcal{C}}(X)$ of smooth superfunctions defined on X. First, we have the L^2 pairing with defined by the measure μ:

$$\prec f, g \succ := \int_X d\mu \bar{f} g \quad . \tag{3.24}$$

Then, we have the L^2 pairing defind by the measure $\mu_\epsilon = \mu\epsilon$:

$$\prec f, g \succ_\epsilon = \int d\mu\ \epsilon \bar{f} g \quad . \tag{3.25}$$

Finally, the Berezin symbol space $\Sigma \subset \underline{\mathcal{C}}(X)$ carries the Berezin superscalar product:

$$\prec f, g \succ_B = \int \bar{f} \diamond g = \langle Q(f), Q(g) \rangle_{HS} = \int_X d\mu(x)\epsilon(x)(\bar{f} \diamond g)(x) \quad . \tag{3.26}$$

3.5 Generalized Toeplitz quantization

Let us now consider the case in which the superscalar product $(\, ,\,)$ on $E = H^0(L)$ is determined by a measure μ on X and a Hermitian form h on L. Since the L^2 Hermitian pairing induced by (μ, h) on $\Gamma(L)$ need not be nondegenerate or positive-definite, we cannot use orthogonal projectors from that space onto the subspace $E = H^0(L)$ of holomorphic supersections. Instead, we return to the definition of Rawnsley supercoherent projectors, which we extend as follows. For every $x \in X$, consider the $\underline{\mathcal{C}}_x$-supermodule $\Gamma_x := \underline{\mathcal{C}}_x \otimes_{\mathbb{C}} E$, which contains $E_x = \underline{\mathcal{O}}_x \otimes_{\mathbb{C}} E$ as a sub-supermodule via the inclusion $\underline{\mathcal{O}}_x \subset \underline{\mathcal{C}}_x$. As in section 3.1., we consider the unique sesquilinear extension of the super Hermitian form $(\, ,\,)$ from E to the supermodule Γ_x. This is given again by eq. (3.2), where now α, β belong to $\underline{\mathcal{C}}_x$, and we again denote the extended form by $(\!(\, ,\,)\!)_x$. This non-degenerate Hermitian form makes Γ_x into a Hermitian supermodule, and we can define the *extended supercoherent projector* $\Pi_x \in \mathrm{End}_{\underline{\mathcal{C}}_x}(\Gamma_x)$ by copying equation (3.5):

$$\Pi_x(s) = \frac{e_q (\!(e_q, s)\!)_x}{(\!(e_q, e_q)\!)_x} \in l_x \quad (s \in \Gamma_x) \quad . \tag{3.27}$$

These extended projectors are even and $\underline{\mathcal{C}}_x$-linear, and their restriction to the sub-supermodule E_x recover Rawnsley's projectors:

$$\Pi_x|_{E_x} = P_x$$

Copying equation (3.22), we define an $\underline{\mathcal{C}}(X)$-linear even operator Π on the $\underline{\mathcal{C}}(X)$-supermodule $\Gamma(E)$ via:

$$\Pi := \int_X d\mu(x)\epsilon(x)\Pi_x \quad . \tag{3.28}$$

Then it is easy to check that Π is an idempotent operator, i.e. $\Pi^2 = \Pi$ and that $\mathrm{im}\,\Pi = E$.

We are now ready to consider Toeplitz quantization. For every smooth superfunction $f \in \underline{\mathcal{C}}(X)$, define the corresponding Toeplitz operator $T_f := T(f) \in \mathrm{End}(E)$ by:

$$T(f)(s) = \Pi(fs) \quad \forall s \in E \quad . \tag{3.29}$$

Using (3.28), this gives:

$$T(f) = \int_X d\mu(x)\epsilon(x) f(x) P_x \quad . \tag{3.30}$$

The underlying map $T : \underline{\mathcal{C}}(X) \to \mathrm{End}(E)$ will be called the generalized Toeplitz quantization of (L, μ, h). As for ordinary manifolds, it satisfies:

$$T(\bar{f}) = T(f)^\dagger \quad \text{and} \quad T(1_X) = \mathrm{id}_E \ . \tag{3.31}$$

Contrary to Berezin quantization, which depends only on the superscalar product $(\, ,\,)$ on E, $T(f)$ depends essentially on the measure μ_ϵ, which is only constrained by the completeness relation (3.22).

3.6 Relation between generalized Berezin and Toeplitz quantization

For ordinary manifolds, the generalized Berezin quantization $Q := \sigma^{-1}$ with respect to the superscalar product $(\, ,\,)$ on E and the generalized Toeplitz quantization T with respect to an integral representation (L, h, μ) of this superscalar product are linked via the *generalized Berezin transform*. The same holds in the case of supermanifolds, as we will show. The map:

$$\beta := \sigma \circ T\,, \tag{3.32}$$

where σ is the Berezin symbol map and T is the Toeplitz quantization map, is called the *generalized Berezin transform* and we have the integral representation:

$$\beta(f)(x) = \int_X d\mu(y)\ \epsilon(y)\Psi(x,y)f(y)\ \ , \tag{3.33}$$

where Ψ is the squared two-point superfunction (3.14). We now have $T(f) = Q(\beta(f))$ and, after restricting to Σ, we find the commutative diagram of bijections:

$$\begin{array}{rcl} \Sigma & \xrightarrow{T|_\Sigma} & \mathrm{End}(E) \\ \beta|_\Sigma \downarrow & & \| \\ \Sigma & \xrightarrow{Q} & \mathrm{End}(E) \end{array}$$

where β and T depend on the measure μ_ϵ but Q and Σ depend only on the superscalar product $(\, ,\,)$. Altogether, Toeplitz quantizations associated with different integral representations of the superscalar product $(\, ,\,)$ on E give different integral descriptions of the Berezin quantization Q defined by this product. Each Toeplitz quantization is equivalent with Q via the corresponding Berezin transform.

Remarks. Let $\langle , \rangle_{\mathrm{HS}}$ be the Hilbert-Schmidt pairing on $\mathrm{End}(E)$ and $\prec,\ \succ_{\mu_\epsilon}$ be the natural super Hermitian pairing on $\underline{\mathcal{C}}(X)$ induced by the measure μ_ϵ. As for ordinary manifolds [10], we have

$$\langle T(f), C\rangle_{\mathrm{HS}} = \mathrm{tr}(T(f)^\dagger C) = \mathrm{tr}(T(\bar f)C) = \int_X d\mu(x)\epsilon(x)\bar f(x)\sigma(C) = \prec f, \sigma(C) \succ_{\mu_\epsilon}\ \ ,$$

which shows that T and σ are adjoint to each other. It follows immediately that T is surjective, since σ is injective and the Berezin transform is a super Hermitian operator with image Σ.

3.7 Changing the superscalar product in generalized Toeplitz quantization

Let us now analyze what happens when we change the superscalar product $(\, ,\,)$ on E to $(\, ,\,)'$ with $(s,t)' := (As, t)$. Equations (3.17) and (3.22) give:

$$\int_X d\mu(x)\epsilon(x)\sigma(A^{-1})(x)P'_x = A^{-1}\ \ , \tag{3.34}$$

Using relations (3.15), (3.16) and (3.21) we find that the epsilon superfunction of the pair $(h, (\, ,\,)')$ is given by:

$$\epsilon'(x) = \epsilon(x)\sigma(A^{-1})\ \ , \tag{3.35}$$

so (3.34) takes the form:

$$\int_X d\mu(x)\epsilon'(x)P'_x = A^{-1} \quad .$$

We can define a new Toeplitz quantization map according to:

$$T'(f) := \int_X d\mu(x)\epsilon'(x)f(x)P'_x \ ,$$

which satisfies $T'(f)^{\oplus} = T(\bar{f})$ as well as:

$$\mathrm{tr}(AT'(f)) = \int_X d\mu(x)\epsilon(x)f(x) = \mathrm{tr}(T(f))$$

and:

$$T'(1_X) = A^{-1} \quad .$$

As on ordinary manifolds, a modified Berezin transform connects generalized Berezin and Toeplitz quantizations with respect to the superscalar product $(\ ,\ )'$, cf. [10].

3.8 Extension to powers of L

The constructions of this supersection can be extended straightforwardly by replacing the very ample super line bundle L with any of its positive powers $L^{\otimes k}$, $k \geq 1$. The new Hermitian superscalar product $(\ ,\ )_k$ on the supervector spaces $E_k := H^0(L^k)$ yields new supercoherent states $e_x^{(k)} \in E_k$ and the associated Rawnsley projectors $P_x^{(k)}$. The latter in turn define injective Berezin symbol maps $\sigma_k : \mathrm{End}(E_k) \to \underline{\mathcal{C}}(X)$ whose images we denote by Σ_k; the inverse of σ_k after corestriction to Σ_k is again denoted by Q_k. Note that the construction depends essentially on the sequence $(\ ,\ )_k$ chosen on the spaces E_k.

4 Special cases: Berezin, Toeplitz and Berezin-Bergman quantization

In this section, we first discuss the classical Berezin and Toeplitz quantization of Hodge supermanifolds using the natural supermetric associated to the Kähler polarization. After this general discussion, we give the quantizations of affine and projective complex superspaces. Not surprisingly, this is quite similar to the case of ordinary Hodge manifolds [10]. We also give a brief discussion of the superanalogue of Berezin-Bergman quantization.

4.1 Classical Berezin and Toeplitz quantization

Given a prequantized Hodge supermanifold (X, ω, L, h), we fix an integer $k_0 > 0$ such that L^{k_0} is very ample. For every integer $k \geq k_0$, endow L^k with the Hermitian supermetric $h_k := h^{\otimes k}$ and consider $E_k := H^0(L^k)$ together with the L^2-scalar product obtained from h_k and the Liouville measure μ_ω.

With these choices, the generalized quantization procedure yields a bijective symbol map $\sigma_k : \mathrm{End}(E_k) \to \Sigma_k \subset \underline{\mathcal{C}}(X)$ and its inverse, the quantization map $Q_k : \Sigma_k \to \mathrm{End}(E_k)$. Moreover, we have the surjective Toeplitz quantization map $T_k : \underline{\mathcal{C}}(X) \to \mathrm{End}(E_k)$. Both are linked by the surjective Berezin transform $\beta_k = \sigma_k \circ T_k : \underline{\mathcal{C}}(X) \to \Sigma_k$ via $T_k = Q_k \circ \beta_k$. Altogether, we have the commutative diagram of bijections:

$$\begin{array}{rcccl} & \Sigma_k & \xrightarrow{T_k|_\Sigma} & \mathrm{End}(E_k) \\ \beta_k|_\Sigma \downarrow & & & \| \\ & \Sigma_k & \xrightarrow{Q_k} & \mathrm{End}(E_k) \end{array}$$

4.2 Relations with deformation quantization and geometric quantization

For ordinary Hodge manifolds, it is possible to show that Toeplitz quantization gives rise to a formal star product leading to deformation quantizations, see [27–29]. Introducing a formal Berezin transform, one can also introduce a corresponding Berezin star product. It should be possible to extend these results to the case of Hodge supermanifolds. For previous work on the deformation quantization of supermanifolds, see [30] in the cases of $U^{1|1}$ and $\mathbb{C}^{m|n}$ via a super-analogue of Toeplitz operators and [31] for split supermanifolds via a Fedosov-type procedure.

As on ordinary manifolds [5, 7], one can define a geometric quantization of a Hodge supermanifold. The prequantization procedure goes back to [6]; a (real) polarization in this context was introduced in [8]. In the case of ordinary manifolds, there is a clear relation between the geometric quantization of quantizable Hermitian symmetric spaces and the Toeplitz quantization procedure as shown in [16]. A similar relationship can be expected for Hodge supermanifolds.

As both these points would take us too far away from the main direction of this work, we refrain from going into more detail.

The Berezin product or supercoherent state star product. The operator product $\diamond_k : \Sigma_k \times \Sigma_k \to \Sigma_k$ introduced in section 3.2,

$$f \diamond_k g \;:=\; \sigma_k(Q_k(f)Q_k(g))\,, \quad f, g \in \Sigma_k\,, \tag{4.1}$$

is also called the *supercoherent state star product*, since $\sigma_k(C) = \mathrm{tr}(CP_x^{(k)})$ is determined by the supercoherent states. It is associative by definition and $(\Sigma_k, \diamond_k, ^-)$ is isomorphic as a $*$-superalgebra to $(\mathrm{End}(E_k), \circ, \dagger)$, an isomorphism being provided by the Berezin quantization map Q_k. As for ordinary manifolds, this is *not* a formal star product, cf. [10].

As an example, consider the Berezin quantization of $(\mathbb{P}^{m|n}, \omega_{\mathrm{FS}})$ with the prequantum super line bundle H^k, where H is again the hyperplane superbundle. If we normalize the homogeneous coordinates $(Z^I) = (z^i, \zeta^\iota) = (z^0, \ldots, z^m, \zeta^1, \ldots, \zeta^n)$ on $\mathbb{P}^{m|n}$ by demanding that $|Z| = 1$, we obtain the particularly simple form [13]:

$$f \diamond_k g = \sum_{I_1,\ldots,I_k} \left(\frac{1}{k!} \frac{\partial}{\partial Z^{I_1}} \cdots \frac{\partial}{\partial Z^{I_k}} f \right) \left(\frac{1}{k!} \frac{\partial}{\partial \bar{Z}^{I_1}} \cdots \frac{\partial}{\partial \bar{Z}^{I_k}} g \right) \,.$$

Using the embedding $\mathbb{P}^{m|n} \hookrightarrow \mathbb{R}^{m^2+n^2-1|2mn}$, one can rewrite this Berezin product as a finite sum of real differential operators, resembling the first terms in an expansion of a formal star product, see e.g. [32].

4.3 The quantization of complex affine superspaces

As one might expect, the Bargmann construction for the quantization of affine space can be extended to the case of affine superspace. Again, we have to replace the space of holomorphic supersections of the quantum super line bundle with the space of supersections which are square integrable with respect to a weighted version of the Liouville measure.

Consider a complex supervector space $V = V_0 \oplus V_1$ of dimension $(m|n)$ over $\mathbb{C}$. While V itself is not a supermanifold, we have the associated supermanifold $\mathbb{A}_V := (V_0, \mathcal{O}_{V_0} \otimes \wedge^\bullet V_1)$ cf. e.g. [33], and in our case $\mathbb{C}^{m|n} := \mathbb{A}_V = (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m}[\zeta^1, \ldots, \zeta^n])$. The structure sheaf of $\mathbb{A}_V$ is freely generated by m even and n odd generators $Z^I = (z^1, \ldots, z^m, \zeta^1, \ldots, \zeta^n)$. We denote by B the algebra of polynomials in these generators, and for any $f \in B$ we write

$$f = \sum_{|\mathbf{p}|=\text{bounded}} a_{\mathbf{p}} \chi_{\mathbf{p}} \,, \tag{4.2}$$

where $|\mathbf{p}| = \sum_{i=1}^{m+n} p_i$, $p_i \in \mathbb{N}$ for $1 \leq i \leq m$ and $p_i \in \{0,1\}$ for $m+1 \leq i \leq n$. The monomials $\chi_{\mathbf{p}}$ are defined as

$$\chi_{\mathbf{p}} := Z^{\mathbf{p}} := (z^1)^{p_1} \ldots (z^m)^{p_m} (\zeta^1)^{p_{m+1}} \ldots (\zeta^n)^{p_{m+n}} \,. \tag{4.3}$$

As mentioned in section 2.2, this space comes with the standard flat Hermitian supermetric whose Kähler form is

$$\omega = \frac{\mathrm{i}}{2\pi} \left(\sum_{i=1}^{m} dz^i \wedge d\bar{z}^i - i \sum_{\iota=1}^{n} d\zeta^\iota \wedge d\bar{\zeta}^\iota \right) = \omega_{IL}\, dz^I \wedge dz^L \tag{4.4}$$

and an associated Liouville measure which is given by the integral form[5]

$$\begin{aligned} d\mu(Z) &:= \frac{1}{(2\pi)^n} |\mathrm{sdet}\,(\omega_{IL})| dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^m \wedge d\bar{z}^m i d\zeta^1 d\bar{\zeta}^1 \ldots i d\zeta^n d\bar{\zeta}^n \\ &= \frac{1}{(2\pi)^m} dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^m \wedge d\bar{z}^m i d\zeta^1 d\bar{\zeta}^1 \ldots i d\zeta^n d\bar{\zeta}^n \,. \end{aligned} \tag{4.5}$$

The Kähler form is polarized with respect to the trivial super line bundle $O := \mathbb{A}_V \times \mathbb{C}$. To obtain a quantum super line bundle, we endow O with the Hermitian supermetric h given by

$$\hat{h}(Z) := e^{-|Z|^2}, \quad |Z|^2 = \sum_{i=1}^{m} \bar{z}^i z^i + i \sum_{\iota=1}^{n} \bar{\zeta}^\iota \zeta^\iota \,, \tag{4.6}$$

which corresponds to the Kähler potential $K(Z) := -\log \hat{h}(Z) = |Z|^2$. We thus have a corresponding L^2-scalar product

$$\langle f, g \rangle_B := \int_{\mathbb{C}^{m|n}} d\mu(Z) e^{-|Z|^2} \bar{f}(Z) g(Z) \tag{4.7}$$

with the normalization $\langle s_0, s_0 \rangle = 1$, where $s_0 = 1$ is the unit constant superfunction on $\mathbb{C}^{m|n}$. We identify now the Bargmann space $\mathcal{B}(\mathbb{C}^{m|n})$ with the space of square integrable holomorphic supersections of O (which contains B as a dense subset). This space

[5]Note that here and in the following $f(Z)$ specifies an arbitrary complex superfunction, not necessarily holomorphic in the Z^I. In physicists' notation, one would write $f(Z, \bar{Z})$.

carries a representation of the Heisenberg superalgebra with m even and n odd pairs of creation/annihilation operators:

$$
\begin{aligned}
(\hat{a}_i^\dagger f)(Z) &:= z^i f(Z)\,,\ (\hat{a}_i f)(Z) := \frac{\partial}{\partial z^i} f(Z)\,,\\
(\hat{\alpha}_\iota^\dagger f)(Z) &:= \zeta^\iota f(Z)\,,\ (\hat{\alpha}_\iota f)(Z) := \frac{\partial}{\partial \zeta^\iota} f(Z)\,,
\end{aligned}
\tag{4.8}
$$

or, summarizing them according to $(\hat{A}_I) = (\hat{a}_i, \hat{\alpha}_\iota)$:

$$
(\hat{A}_I^\dagger f)(Z) := Z^I f(Z)\,, \quad (\hat{A}_I f)(Z) := \frac{\partial}{\partial Z^I} f(Z)\ .
$$

These operators satisfy the commutation relations[6]

$$
[\hat{a}_i, \hat{a}_j^\dagger] := [\hat{a}_i, \hat{a}_j^\dagger]_- = \delta_{ij}\,, \quad \{\hat{\alpha}_\iota, \hat{\alpha}_\gamma^\dagger\} := [\hat{\alpha}_\iota, \hat{\alpha}_\gamma^\dagger]_+ = \mathrm{i}\delta_{\iota\gamma}\,,
\tag{4.9}
$$

or, using the supercommutator $\{\!\![\,,\,]\!\!\}$:

$$
\{\!\![\hat{A}_I, \hat{A}_J^\dagger]\!\!\} = \mathrm{i}^{\tilde{I}\tilde{J}} \delta_{IJ}\ .
\tag{4.10}
$$

We normalize the vacuum vector in $\mathcal{B}(\mathbb{C}^{m|n})$ to the constant unit function $|0\rangle := 1$, and setting $\langle 0|0\rangle_B = 1$ yields together with the commutation relations (4.9) a Hermitian superscalar product $\langle\ |\ \rangle_B$ on $\mathcal{B}$. The normalized occupation vectors are given by:

$$
|\mathbf{p}\rangle = \frac{1}{\sqrt{\mathbf{p}!}}\chi_{\mathbf{p}} = \frac{(\hat{A}^\dagger)^{\mathbf{p}}}{\sqrt{\mathbf{p}!}}|0\rangle \quad \text{with} \quad ||\chi_{\mathbf{p}}||_B^2 = (i)^{(\sum_{\iota=1}^n p_\iota) \bmod 2}\mathbf{p}! = (i)^{\widetilde{|\mathbf{p}\rangle}}\mathbf{p}!\,,
\tag{4.11}
$$

where $\mathbf{p}! := p_0! \dots p_n!$. Defining the number operators $\hat{N}_I := (-i)^{\tilde{I}} \hat{A}_I^\dagger \hat{A}_I$, we have $\hat{N}_I|\mathbf{p}\rangle = p_I|\mathbf{p}\rangle$. The total number operator

$$
\hat{N} := \sum_{I=1}^{m+n} \hat{N}_I = \sum_{i=1}^m \hat{a}_i^\dagger \hat{a}_i - i \sum_{\iota=1}^n \hat{\alpha}_\iota^\dagger \hat{\alpha}_\iota
\tag{4.12}
$$

allows us to introduce the decomposition $\mathcal{B} := \overline{\oplus}_{k=0}^\infty B_k$ with $B_k = \ker(\hat{N} - k)$.

We define the supercoherent vectors with respect to $q = s_0(z) = 1 \in \mathcal{O}_z$ and this definition yields the usual super Glauber vectors:

$$
\begin{aligned}
|Z\rangle = e^{\sum_{I=1}^{m+n} i^{\tilde{I}} \bar{Z}^I \hat{A}_I^\dagger}|0\rangle &= e^{\sum_{i=1}^m \bar{z}^i \hat{a}_i^\dagger + i\sum_{\iota=1}^n \bar{\zeta}^\iota \hat{\alpha}_\iota^\dagger}|0\rangle\,,\\
|Z\rangle &= \sum_{\mathbf{p}} (i)^{\widetilde{|\mathbf{p}\rangle}} \frac{\bar{Z}^{\mathbf{p}}}{\sqrt{\mathbf{p}!}}|\mathbf{p}\rangle\ ,\\
\hat{A}_I|Z\rangle &= \bar{Z}^I|Z\rangle\,,\\
\langle Z_1|Z_2\rangle_B &= e^{(Z_2, Z_1)}\,,
\end{aligned}
\tag{4.13}
$$

[6]The factor of i is necessary to match our conventions for complex conjugation of objects of odd parity: $(\hat{\alpha}_\alpha \hat{\alpha}_\beta)^\dagger = -\hat{\alpha}_\beta^\dagger \hat{\alpha}_\alpha^\dagger$.

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where $\bar{Z}^{\mathbf{p}} = \bar{z}_1^{p_1} \dots \bar{z}_n^{p_m} \bar{\zeta}_1^{p_{m+1}} \dots \bar{\zeta}_n^{p_{m+n}}$ and $(Z_1, Z_2) := \sum_{i=1}^m \bar{z}_1^i z_2^i + i \sum_{\iota=1}^n \bar{\zeta}_1^\iota \zeta_2^\iota$ denotes the superscalar product on $\mathbb{C}^{m|n}$. We have as usual

$$f(Z) = \langle Z|f\rangle_B \quad \text{for} \quad f \in \mathcal{B} \ .$$

The reproducing kernel is the super Bergman kernel:

$$K_B(Z_1, Z_2) = \frac{\langle Z_1|Z_2\rangle}{\sqrt{\langle Z_1|Z_1\rangle\langle Z_2|Z_2\rangle}} = e^{-\frac{1}{2}(|Z_1|^2+|Z_2|^2)+(Z_2,Z_1)} \ . \tag{4.14}$$

The Rawnsley projector is given by

$$P_Z = \frac{1}{\langle Z|Z\rangle_B}|Z\rangle\langle Z|_B = e^{-|Z|^2}|Z\rangle\langle Z|_B \tag{4.15}$$

with constant epsilon superfunction $\epsilon_{\mathbb{C}^{m|n}}(Z) = \hat{h}(Z)\langle Z|Z\rangle_B = 1$ and decomposition of the identity $\int_{\mathbb{C}^{m|n}} d\mu(Z) P_Z = \mathbb{1}_{\mathcal{B}}$.

Toeplitz quantization of $\mathbb{A}_V$. The Toeplitz quantization of $f \in \underline{\mathcal{C}}(\mathbb{C}^{m|n})$ is given by:

$$T(f) = \int_{\mathbb{C}^{m|n}} d\mu(Z) f(Z) P_Z = \int_{\mathbb{C}^{m|n}} d\mu(Z) e^{-|Z|^2} f(Z)|Z\rangle\langle Z|_B \ . \tag{4.16}$$

In particular, we have $T(Z^I) = \hat{A}_I^\dagger$ and $T(\bar{Z}^I) = \hat{A}_I$. When f is a polynomial in Z and $\bar{Z}$, (4.16) obviously reduces to the anti-Wick prescription:

$$T(f(Z,\bar{Z})) = \vdots f(\hat{A}^\dagger, \hat{A}) \vdots \ ,$$

where the triple dots indicate antinormal ordering. In this case, T is not surjective due to the infinite-dimensionality of the Bargmann space.

Berezin quantization of $\mathbb{A}_V$. The Berezin symbol map is easily extended as well. It is defined on the algebra $\mathcal{L}(\mathcal{B})$ of bounded operators in the Bargmann space and maps them into $\underline{\mathcal{C}}(\mathbb{C}^{m|n})$ as follows:

$$\sigma(C)(Z) = e^{-|Z|^2}\langle Z|C|Z\rangle_B \ .$$

The Berezin transform $\beta(f) = \sigma \circ T$ is given by:

$$\beta(f)(Z_1) = \int_{\mathbb{C}^{m|n}} d\mu(Z_2) f(Z_2) e^{-|Z_1-Z_2|^2} \ .$$

The symbol map gives rise to the Berezin quantization map $Q : \Sigma \to \mathcal{L}(\mathcal{B})$, where $\Sigma \subset \underline{\mathcal{C}}(\mathbb{C}^{m|n})$ is the image of σ. We have $Q(Z^I) = \hat{A}_I^\dagger$ and $Q(\bar{Z}^I) = \hat{A}_I$. For a polynomial superfunction $f(Z,\bar{Z})$, we find:

$$Q(f) =: f(\hat{A}^\dagger, \hat{A}) : \ ,$$

where the double dots indicate normal ordering. Hence both quantization prescriptions send Z^I into $\hat{A}_I^\dagger$ and $\bar{Z}^I$ into $\hat{A}_I$, but Toeplitz quantization corresponds to anti-Wick ordering, while Berezin quantization corresponds to Wick ordering.

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Restricted supercoherent vectors. For later use, consider the expansion of Glauber's supercoherent vectors $|Z\rangle$ in components $|Z,k\rangle$ of fixed total particle number k, i.e. $\hat{N}|Z,k\rangle = k$:

$$|Z\rangle = \sum_{k=0}^{\infty} |Z,k\rangle\,, \quad |Z,k\rangle := \frac{1}{k!}\left(\sum_{i=1}^{m} \bar{z}^i \hat{a}_i^\dagger + i \sum_{\iota=1}^{n} \bar{\zeta}^\iota \hat{\alpha}_\iota^\dagger\right)^k |0\rangle\ .$$

We note for future reference that $\langle Z,k|Z,k\rangle_B = \frac{1}{k!}|Z|^{2k}$ and $\hat{A}_I|Z,k\rangle = \bar{Z}^I|Z,k-1\rangle$. Note furthermore that $|\lambda Z,k\rangle = \bar{\lambda}^k|Z,k\rangle$ for any $\lambda \in \mathbb{C}$, and therefore the ray $\mathbb{C}^*|Z,k\rangle$ depends only on the image $[Z]$ of Z in the projective superspace $\mathbb{P}^{m-1|n}$.

4.4 The quantization of complex projective superspaces

It is now easy to carry out the quantization of complex projective superspaces. For earlier discussions of these spaces relying on group theoretic methods, see [13, 34]. Another possible approach would be to extend the techniques of [35, 36] to the supercase.

Consider the supermanifold $\mathbb{P}^{m|n}$ as introduced in section 2.2 with homogeneous supercoordinates $Z^I = (z^0, \ldots, z^m, \zeta^1, \ldots, \zeta^n)$. As a quantum super line bundle, we take the super hyperplane bundle $H := O(1)$, which is very ample. The space of supersections $H^0(H^k)$ is the space of homogeneous polynomials of degree k and can thus be identified with $B_k \in \mathcal{B}$, where $\mathcal{B}$ is the Bargmann space used in the quantization of $\mathbb{C}^{m+1|n}$. Notice that:

$$\begin{aligned}
\dim B_k &= (b_k^0|b_k^1)\,,\\
b_k^0 &= \sum_{i=0}^{[\min\{k,n\}/2]} \frac{(m+1+(k-2i))!}{(m+1)!(k-2i)!}\frac{n!}{(n-2i)!(2i)!}\,,\\
b_k^1 &= \sum_{i=0}^{[(\min\{k,n\}-1)/2]} \frac{(m+1+(k-(2i+1))!}{(m+1)!(k-(2i+1))!}\frac{n!}{(n-(2i+1))!(2i+1)!}\,,
\end{aligned} \tag{4.17}$$

where b_k^0 and b_k^1 are the even and odd dimensions of B_k, respectively, and $[\ldots]$ denotes taking the integral part. The first factor in the sums corresponds to the symmetrized even homogeneous coordinates z^i, while the second factor represents the antisymmetrized odd homogeneous coordinates ζ^ι.

We endow the hyperplane superbundle H with the Hermitian supermetric given by

$$h_{\mathrm{FS}}([Z])(Z^I, Z^I) = \frac{|Z^I|^2}{|Z|^2}\,, \tag{4.18}$$

which is associated to the following Kähler supermetric on $\mathbb{P}^{m|n}$:

$$\omega_{\mathrm{FS}}([Z]) = \frac{i}{2\pi}\partial\bar{\partial}\log|Z|^2 = \frac{i}{2\pi}\partial\bar{\partial}\log(1+|Z_0|^2)\ , \tag{4.19}$$

cf. section 2.2. Let us be more explicit and restrict to the patch U_0 for which $z^0 \neq 0$ with local coordinates $(Z_0^I) = (z_0^i, \zeta_0^\iota)$, $I = 1, \ldots, m+n$, where $z_0^i = \frac{z^i}{z^0}$ and $\zeta_0^\iota = \frac{\zeta^\iota}{z^0}$. On this patch, the Kähler form reads as

$$\omega_{\mathrm{FS}}|_{U_0} = \omega_{IL}\, dZ_0^I \wedge dZ_0^L \tag{4.20}$$

with

$$\omega_{IL} = \left(\begin{array}{c|c} \omega_{il} & \omega_{i\lambda} \\ \hline \omega_{\iota l} & \omega_{\iota\lambda} \end{array}\right) = \frac{\mathrm{i}}{2\pi(1+|Z_0|^2)^2} \left(\begin{array}{c|c} \delta^{il}(1+|Z_0|^2) - \bar{z}_0^i z_0^l & -\mathrm{i}\bar{z}_0^i \zeta_0^\lambda \\ \hline -\mathrm{i}\bar{\zeta}_0^\iota z_0^l & \mathrm{i}\delta^{\iota\lambda}(1+|Z_0|^2) - \bar{\zeta}_0^\iota \zeta_0^\lambda \end{array}\right) .$$

The corresponding Liouville measure $d\mu(Z)$ is given in the coordinates on the patch U_0 as

$$(2\pi)^n |\mathrm{sdet}\,(\omega_{IL})| dz_0^1 \wedge d\bar{z}_0^1 \wedge \ldots \wedge dz_0^m \wedge d\bar{z}_0^m \mathrm{i}d\zeta_0^1 d\bar{\zeta}_0^1 \ldots \mathrm{i}d\zeta_0^n d\bar{\zeta}_0^n , \tag{4.21}$$

where

$$|\mathrm{sdet}\,(\omega_{IL})| := \frac{\det((\omega_{il}) - (\omega_{i\lambda})(\omega_{\iota\lambda})^{-1}(\omega_{\iota l}))}{\det(\omega_{\iota\lambda})} = (2\pi)^{n-m}(1+|Z_0|^2)^{n-m-1} . \tag{4.22}$$

Note that the volume of $\mathbb{P}^{m|n}$ vanishes, if $(n-m-1) \geq 0$ because of the Berezin integration in the measure. Otherwise, we can use the formula

$$\frac{1}{(1+\sum_i \bar{z}_0^i z_0^i + i\sum_\iota \bar{\zeta}_0^\iota \zeta_0^\iota)^g} = \sum_{\ell=0}^{n} (-1)^\ell \frac{(g-1+\ell)!}{(g-1)!\ell!} \frac{1}{(1+\sum_i \bar{z}_0^i z_0^i)^{g+\ell}} \left(i \sum_\iota \bar{\zeta}_0^\iota \zeta_0^\iota \right)^\ell \tag{4.23}$$

and arrived at the closed expression

$$\mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^{m|n}) = \frac{1}{m!} \frac{m!}{(m-n)!} , \tag{4.24}$$

and in particular, we have $\mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^{m|0}) = \frac{1}{m!}$.

The supermetric on the hyperplane superbundle H extends to the tensor product supermetric $h^k_{\mathrm{FS}} := h^{\otimes k}_{\mathrm{FS}}$, which satisfies:

$$h^k_{\mathrm{FS}}([Z])(S([Z]), S([Z])) = \frac{|s(Z)|^2}{|Z|^{2k}} \tag{4.25}$$

for all supersections $S \in H^0(H^k)$ and their corresponding $s \in B_k$. The space $H^0(H^k) \cong B_k$ carries the associated L^2-product:

$$\langle s_1, s_2 \rangle_k := \langle S_1, S_2 \rangle_k^{h^k_{\mathrm{FS}}} = \int_{\mathbb{P}^{m|n}} d\mu(z)\ h^k_{\mathrm{FS}}(S_1, S_2) \quad . \tag{4.26}$$

Note that the monomials $\chi_{\mathbf{p}}$ with $|\mathbf{p}| = k$ provide an orthogonal but not orthonormal basis of B_k with respect to the superscalar product (4.26). Using formula (4.23), we easily verify that

$$\langle s, t \rangle_k = \frac{1}{(m-n+k)!} \langle s, t \rangle_B \quad \forall s, t \in B_k \quad . \tag{4.27}$$

The quantization of $\mathbb{P}^{m|n}$ proceeds now in a straightforward manner. The supercoherent states of the quantum super line bundle H^k are the Glauber supercoherent states restricted at level k and from these, we construct the supercoherent projectors

$$P^{(k)}_{[Z]} := \frac{|Z,k\rangle\langle Z,k|_B}{\langle Z,k|Z,k\rangle_B} . \tag{4.28}$$

The overcompleteness relation takes the form

$$(b_k^0 - b_k^1) \int_{\mathbb{P}^{m|n}} d\mu([Z]) P_{[Z]}^{(k)} = \mathrm{vol}(\mathbb{P}^{m|n}) P_k \ , \tag{4.29}$$

where P_k is the orthoprojector on B_k in $\mathcal{B}(\mathbb{C}^{m|n})$. The normalization is obtained by taking the supertrace of both sides. Note that interestingly whenever $m < n$ and thus $\mathrm{vol}(\mathbb{P}^{m|n}) = 0$, we also have $b_k^0 - b_k^1 = 0$ for $k > 0$ as one can show e.g. by complete induction. Therefore, this normalization condition does not give any additional constraint in these cases. Alternatively, we can calculate the ordinary trace. While $\mathrm{str}\, P_{[Z]}^{(k)} = 1$ and $\mathrm{str}\, P_k = b_0^k - b_1^k$, we have

$$\mathrm{tr}\,(P_{[Z]}^{(k)}) = \left(\frac{\sum_{i=0}^m \bar{z}^i z^i - i \sum_{\iota=1}^n \bar{\zeta}^\iota \zeta^\iota}{\sum_{i=0}^m \bar{z}^i z^i + i \sum_{\iota=1}^n \bar{\zeta}^\iota \zeta^\iota} \right)^k \ , \quad \mathrm{tr}\, P_k = b_0^k + b_1^k \ . \tag{4.30}$$

The expression

$$\mathrm{vol}_{\mathrm{tr}}(\mathbb{P}^{m|n}) := \int_{\mathbb{P}^{m|n}} d\mu([Z]) \left(\frac{\sum_{i=0}^m \bar{z}^i z^i - i \sum_{\iota=1}^n \bar{\zeta}^\iota \zeta^\iota}{\sum_{i=0}^m \bar{z}^i z^i + i \sum_{\iota=1}^n \bar{\zeta}^\iota \zeta^\iota} \right)^k \tag{4.31}$$

is clearly non-vanishing and can easily be evaluated in every concrete case. Our new normalization of the overcompleteness relation (4.29) now reads as

$$\frac{b_k^0 + b_k^1}{\mathrm{vol}_{\mathrm{tr}}(\mathbb{P}^{m|n})} \int_{\mathbb{P}^{m|n}} d\mu([Z]) P_{[Z]}^{(k)} = P_k \ . \tag{4.32}$$

We will restrict ourselves to superfunctions on $\mathbb{P}^{m|n}$ of the form:

$$f_{\mathcal{I}\mathcal{J}}([Z]) := \frac{\bar{Z}^{\mathcal{I}} Z^{\mathcal{J}}}{|Z|^{2k}} := \frac{(\bar{z}^0)^{\mathcal{I}_0} \dots (\bar{\zeta}^n)^{\mathcal{I}_{m+n}} (z^0)^{\mathcal{J}_0} \dots (\zeta^n)^{\mathcal{J}_{m+n}}}{|Z|^{2k}} \ , \tag{4.33}$$

$f_{\mathcal{I}\mathcal{J}}[z] \in \underline{\mathcal{C}}(\mathbb{P}^{m|n})$, where $\mathcal{I} = (\mathcal{I}_0 \dots \mathcal{I}_{m+n}), \mathcal{J} = (\mathcal{J}_0 \dots \mathcal{J}_{m+n})$ with $\mathcal{I}_L, \mathcal{J}_L \in \mathbb{N}$ for $L \leq m$ and $\mathcal{I}_L, \mathcal{J}_L \in \{0, 1\}$ for $L > m$ and $|\mathcal{I}| = |\mathcal{J}| = k$ and where we set $f_{\mathcal{I}\mathcal{J}} = 1$ for $m = n = 0$. Furthermore, we can decompose $\mathcal{S}(\mathbb{P}^{m|n})$ into the subsets $\mathcal{S}_k(\mathbb{P}^{m|n})$, which are spanned by the superfunctions $f_{\mathcal{I}\mathcal{J}}$ with $|\mathcal{I}| = |\mathcal{J}| = k$; note that $\mathcal{S}_0(\mathbb{P}^{m|n}) = \mathbb{C}$. For any $L = 0 \dots m+n$, let $\Delta_L \in \mathbb{N}^{m+n+1}$ be given by $\Delta_L(I) = \delta_{IL}$. The obvious relation:

$$f_{\mathcal{I}\mathcal{J}} = \sum_{L=0}^{m+n} f_{\mathcal{I}+\Delta_L, \mathcal{J}+\Delta_L}$$

shows that $\mathcal{S}_k(\mathbb{P}^{m|n}) \subset \mathcal{S}_{k+1}(\mathbb{P}^{m|n})$ for all $k \geq 0$, so that $\mathcal{S}(\mathbb{P}^{m|n}) = \cup_{k=0}^\infty \mathcal{S}_k(\mathbb{P}^{m|n})$ is a filtered $*$-algebra generated by the elements $f_{\mathrm{IJ}} = \frac{\bar{Z}_I Z_J}{|Z|^2} \in \mathcal{S}_1(\mathbb{P}^{m|n})$.

Note that the space $\mathcal{S}(\mathbb{P}^{m|n})$ forms a good approximation to $\mathcal{C}^\infty(\mathbb{P}^{m|n})$: Since $\mathbb{P}^{m|n}$ is a split supermanifold, any superfunction $f \in \underline{\mathcal{C}}(\mathbb{P}^{m|n})$ allows for a globally valid expansion of the form

$$f(Z) = \sum_{|\mathcal{A}|+|\mathcal{C}|=|\mathcal{B}|+|\mathcal{D}|} f_{\mathcal{ABCD}}(z) \frac{\zeta^{\mathcal{A}} \bar{\zeta}^{\mathcal{B}} z^{\mathcal{C}} \bar{z}^{\mathcal{D}}}{|Z|^{|\mathcal{A}|+|\mathcal{B}|+|\mathcal{C}|+|\mathcal{D}|}} \tag{4.34}$$

with multi-indices $\mathcal{A}, \mathcal{B}$ and coefficient superfunctions $f_{\mathcal{ABCD}}(z) \in \underline{\mathcal{C}}(\mathbb{P}^{m|0})$; the latter are well approximated, as $\mathcal{S}(\mathbb{P}^{m|0})$, which is contained in $\mathcal{S}(\mathbb{P}^{m|n})$, is dense in $(\mathcal{C}^\infty(\mathbb{P}^m), ||\ ||_\infty)$. The latter is true due to the Stone-Weierstraß theorem, cf. [10]. To define an orthoprojector π_k onto $\mathcal{S}_k(\mathbb{P}^{m|n})$, we cannot rely on an L^2-scalar product on $\mathbb{P}^{m|n}$. We can, however, project each coefficient superfunction $f_{\mathcal{ABCD}}$ with $k - |\mathcal{A}| - |\mathcal{C}| \leq 0$ onto $\mathcal{S}_{k-|\mathcal{A}|-|\mathcal{C}|}(\mathbb{P}^{m|0})$ using the ordinary L^2-orthoprojector on $\mathcal{C}^\infty(\mathbb{P}^m)$ and plug these back into the expansion (4.34). This clearly yields an element of $\mathcal{S}_k(\mathbb{P}^{m|n})$.

Toeplitz quantization of $\mathbb{P}^{m|n}$. We define the Toeplitz quantization map T_k : $\mathcal{C}^\infty(\mathbb{P}^{m|n}, \mathbb{C}) \to \mathrm{End}\,(B_k)$ according to

$$T_k(f) = \frac{b_k^0 + b_k^1}{\mathrm{vol_{tr}}(\mathbb{P}^{m|n})} \int_{\mathbb{P}^{m|n}} d\mu([Z]) f([Z]) P^{(k)}_{[Z]} \quad . \tag{4.35}$$

Due to $\hat{A}_I|Z,k\rangle = \bar{Z}^I|Z,k-1\rangle$, we find:

$$\begin{aligned} T_k(f_{\mathcal{IJ}}) &= \frac{b_k^0 + b_k^1}{\mathrm{vol_{tr}}(\mathbb{P}^{m|n})} \int_{\mathbb{P}^{m|n}} d\mu([Z]) \frac{\hat{A}^{\mathcal{I}}|Z,k+d\rangle\langle Z,k+d|_B(\hat{A}^\dagger)^{\mathcal{J}}}{|Z|^{2(k+m)}} \\ &= \frac{b_k^0 + b_k^1}{\mathrm{vol_{tr}}(\mathbb{P}^{m|n})} \hat{A}^{\mathcal{I}} P_{k+m}(\hat{A}^\dagger)^{\mathcal{J}} \quad , \end{aligned} \tag{4.36}$$

and thus the map $T_k(f)$ is surjective. As a special case, we have:

$$T_k(f_{IJ}) = \frac{b_1^0 + b_1^1}{\mathrm{vol_{tr}}(\mathbb{P}^{m|n})} \hat{A}_I \hat{A}_J^\dagger \quad .$$

Berezin quantization of $\mathbb{P}^{m|n}$. The Berezin symbol map $\sigma_k : \mathrm{End}(B_k) \to \mathcal{C}^\infty(\mathbb{P}^{m|n}, \mathbb{C})$ takes the form:

$$\sigma_k(C)([Z]) = \frac{\langle Z,k|C|Z,k\rangle}{\langle Z,k|Z,k\rangle} \qquad \forall C \in \mathrm{End}(B_k) \quad .$$

This map is injective, and we can define an inverse on $\Sigma_k := \mathrm{im}\,\sigma_k$ which yields the Berezin quantization map $Q_k : \Sigma_k(\mathbb{P}^{m|n}) \to \mathrm{End}(B_k)$, which is a linear isomorphism. Under quantization, the superfunctions (4.33) are mapped to:

$$Q_k(f_{\mathcal{IJ}}) = \frac{1}{k!} P_k (\hat{A}^\dagger)^{\mathcal{I}} \hat{A}^{\mathcal{J}} P_k\,,$$

and we have in particular:

$$Q_k(f_{IJ}) = \hat{A}_J^\dagger \hat{A}_I \quad .$$

Notice that the operators $\hat{f}_{\mathcal{IJ}} = P_k(\hat{A}^\dagger)^{\mathcal{I}}\hat{A}^{\mathcal{J}}P_k$ with $|\mathcal{I}| = |\mathcal{J}| = k$ provide a basis for $\mathrm{End}(B_k)$, and thus the image $\Sigma_k(\mathbb{P}^{m|n})$ can be identified with $\mathcal{S}_k(\mathbb{P}^{m|n})$. Therefore, $\Sigma_k(\mathbb{P}^{m|n})$ provides a weakly exhaustive filtration of $(\mathcal{C}^\infty(\mathbb{P}^{m|n}), ||\ ||_\infty^\circ)$:

$$\overline{\cup_{k=1}^\infty \Sigma_k(\mathbb{P}^{m|n})} = \underline{\mathcal{C}}(\mathbb{P}^{m|n}) \quad .$$

The Berezin transform $\beta_k : \underline{\mathcal{C}}(\mathbb{P}^{m|n}) \to \Sigma_k(\mathbb{P}^{m|n})$ is here defined as:

$$\beta_k(f)([Z]) = \sigma_k(T_k(f)) = \frac{b_k^0 + b_k^1}{\mathrm{vol_{tr}}(\mathbb{P}^{m|n})} \int_{\mathbb{P}^{m|n}} d\mu([Y]) \left(\frac{|(Y,k|Z,k)|}{(Y,k|Y,k)(Z,k|Z,k)} \right)^{2k} \quad .$$

As in the quantization of affine space, Berezin and Toeplitz quantizations use Wick and anti-Wick orderings, respectively.

Remarks. As the space $\mathbb{P}^{m|n}$ is the coset space $\mathrm{U}(m+1|n)/(\mathrm{U}(1|0) \times \mathrm{U}(m|n))$, the Rawnsley supercoherent states can be identified with the Perelomov supercoherent states. Rather obviously, the spaces B_k and $\mathcal{B}_k$ form irreducible representations of the supergroup $\mathrm{U}(m+1|n)$. For further details on the group theoretic aspects of Berezin-quantized $\mathbb{P}^{m|n}$, see [13].

4.5 Berezin-Bergman quantization

In [37], a quantization prescription was proposed for projective algebraic varieties, which relied on their embedding into projective space. More explicitly, the idea was to use the identification of supersections of the quantum bundle $H^0(H^k)$ on $\mathbb{P}^m$ and the Hilbert space B_k in the quantization of the embedded variety X by factoring out an ideal. The zero locus conditions $f_i = 0$ defining $X \subset \mathbb{P}^m$ reducing the space $H^0(H^k)$ would go over into conditions $\hat{f}_i|\mu\rangle = 0$ for all $|\mu\rangle \in B_k$. As shown in [10], this *Berezin-Bergman quantization* corresponds to a generalized Berezin quantization. A similar construction can also be performed in the case of Hodge supermanifolds and we outline this construction in the following.

We start from a polarized complex supermanifold (X, L) with very ample super line bundle L and $\dim_{\mathbb{C}} H^0(L) = (m|n)$. The homogeneous coordinate ring of X is (bi-)graded: $R(X, L) = \oplus_{k=0}^{\infty} H^0(L^k) =: \oplus_{k=0}^{\infty} E_k$, and we have an isomorphism of graded algebras $\phi : R \xrightarrow{\sim} B/I$. Here, B is the graded symmetric algebra $B = \oplus_{k=0}^{\infty} E_1^{\odot k}$ and $I = \oplus_{k=0}^{\infty} I_k$ is a graded ideal in B. The Kodaira superembedding theorem [9], gives a superembedding defined by L in which X is presented as a projective algebraic supervariety in $\mathbb{P}^{m|n}$ with vanishing ideal I. We have

$$I_k \subset B_k \quad \text{and} \quad E_k \simeq B_k/I_k \ . \tag{4.37}$$

At every level k, one has two natural choices for introducing a super Hermitian pairing on $H^0(L^k)$. The first is to take the usual L^2-product:

$$\langle s, t\rangle_k = \int_X d\mu_\omega h^{\otimes k}(s, t) \ ,$$

while the second one is induced from B as follows:

$$(s_1 \odot \dots s_k, t_1 \odot \dots t_l)_B = \frac{1}{k!}\delta_{k,l} \sum_{\sigma \in S_k} \epsilon(\sigma, t_1 \dots t_k)(s_1, t_{\sigma(1)})_1 \dots (s_k, t_{\sigma(k)})_1 \ . \tag{4.38}$$

Here $(\ ,\ )_1$ is the superscalar product on E_1, S_k is the symmetric group on k letters, $s_i, t_i \in E_1$ and $\epsilon(\sigma, t_1 \dots t_k)$ is the Koszul sign in the graded symmetric product. Notice that the second choice is actually the restriction of (4.7) to B_k.

Let $I_k^\perp := \{s \in B_k | (s,t)_B = 0 \ \ \forall t \in I_k\}$ and notice that we can identify this space with E_k. First, we can identify B_k with $H^0(H^k)$, where H is the super hyperplane bundle over $\mathbb{P}^{m|n}$. Then we have a restriction $i_k^* : H^0(H^k) = B_k \to H^0(L^k) = E_k$. As $I_k = \ker i_k^*$ and since i_k^* is surjective, we have an isomorphism $\phi_k := E_k \to I_k^\perp \simeq B_k/I_k$. Then we define a superscalar product $(\ ,\ )_k$ on E_k via:

$$(s,t)_k := \alpha_k(\phi_k(s), \phi_k(t))_B \ . \tag{4.39}$$

For ordinary manifolds, the choice of the normalization constants α_k depended on the volumes of X and $\mathbb{P}^{m|n}$ as well as the dimensions of B_k and E_k. In the supercase, one again has to introduce the trace volume $\mathrm{vol}_{\mathrm{tr}}$ if the classical supervolume of X or $\mathbb{P}^{m|n}$ vanishes. We can then impose the usual condition for a "good" quantization: that under generalized Berezin quantization, the unit superfunction 1_X is mapped into the unit operator $\mathbb{1}$ on E_k.

Definition. The *Berezin-Bergman quantization* of (X, L) determined by the superscalar product $(\ ,\ )_1$ on $H^0(L)$ is the generalized Berezin quantization performed with respect to the sequence of superscalar products $(\ ,\ )_k$ on $H^0(L^k)$ defined in (4.39).

Remarks. As the vanishing ideal I is zero in the case of $\mathbb{P}^{m|n}$, Berezin-Bergman quantization here corresponds to ordinary Berezin quantization.

If I is generated by p homogeneous polynomials $F_1 \dots F_p$ of degrees at least two, then we have

$$I^\perp = \cap_{l=1}^p \ker \bar{F}_l(\hat{A}^\dagger) \ ,$$

where $\bar{F}$ is the polynomial in Z_I obtained by conjugating all *coefficients* of F as in the case of ordinary manifolds, cf. [10].

5 Regularizing supersymmetric quantum field theories

As stated in the introduction, one of the major applications of Berezin-quantized manifolds in physics is the regularization of path integrals and the numerical treatment of quantum field theories. In this section, we extend the definition of fuzzy superscalar field theories, i.e. superscalar field theories defined on Berezin-quantized Hodge manifolds, to some supersymmetric cases. For the earliest work in this direction, see [12]; our discussion will follow along similar lines as those proposed in [38].

It should be clear that an exhaustive discussion of supersymmetric superscalar field theories on quantized supermanifolds, which, as we will see, requires that they admit a superfield description, cannot possibly[7] be performed within this work. We will therefore restrict our discussion to giving an example in more detail: the $\mathcal{N} = (2, 2)$ supersymmetric sigma model on Berezin-quantized $\mathbb{P}^{1|2}$. We will also comment on its topological twist, which can, in principle, be defined on an arbitrary Riemann surface. These theories are particularly interesting, as they serve as the basic building blocks for string theories.

5.1 Fuzzy scalar field theories

Classical (real) scalar field theory on a Kähler manifold (X, ω) is usually given by an action functional of the form

$$S[\phi] = \frac{1}{\mathrm{vol}_\omega(X)} \int_X \frac{\omega^n}{n!} \left(\phi\Delta\phi + V(\phi)\right), \quad \phi \in \mathcal{C}^\infty(X, \mathbb{R}) \, , \tag{5.1}$$

where Δ is the Laplace operator on (X, ω) and $V(\phi)$, the *potential*, is a polynomial in ϕ with real coefficients. To study a similar field theory on a quantized manifold, we need a

[7] cf. e.g. [39] just for the case of two dimensions

quantization of the classical Laplace operator. Two such quantizations are possible and we will briefly review them below. For a more detailed discussion, see [10].

In the following, consider a quantized Hodge manifold (X, ω, E_k) with symbol space $\Sigma_k = \sigma(\mathrm{End}(E_k))$. In general, there are two ways of defining a quantum analogue to an operator $\mathcal{D} : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$. First, we define by $\mathcal{D}_k$ a truncated map $\Sigma_k \to \Sigma_k$ as:

$$\mathcal{D}_k := \pi_k \circ \mathcal{D}|_{\Sigma_k} , \tag{5.2}$$

where π_k is the orthoprojector with respect to the ordinary L^2-scalar product on (X, ω). The Berezin push $\mathcal{D}_k^B$ of an operator $\mathcal{D}$ is then defined as the following map $\mathcal{D}_k^B$: $\mathrm{End}(E_k) \to \mathrm{End}(E_k)$:

$$\mathcal{D}_k^B := Q_k \circ \mathcal{D}_k \circ \sigma_k \ . \tag{5.3}$$

Roughly speaking, the Berezin push of an operator acts as the corresponding operator in the continuum (up to truncations), and we have in particular in the case of the identity operator $\mathcal{D}(f) = f$ for all $f \in \mathcal{C}^\infty(X)$ the quantization $\mathcal{D}_k^B(\hat{f}) = \hat{f}$ for all $\hat{f} \in \mathrm{End}(E_k)$. Hermitian operators with respect to the natural L^2-norm on (X, ω) are, however, not mapped into Hermitian operators with respect to the natural Hilbert-Schmidt norm on $\mathrm{End}(E_k)$.

Alternatively, we can define the *Berezin-Toeplitz lift* of an operator $\mathcal{D} : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ as

$$\hat{\mathcal{D}}_k := T_k \circ M_{\frac{1}{\epsilon_k}} \circ \mathcal{D} \circ \sigma_k , \tag{5.4}$$

$\hat{\mathcal{D}} : \mathrm{End}(E_k) \to \mathrm{End}(E_k)$, where $M_\alpha(f) := \alpha f$, $\alpha, f \in \mathcal{C}^\infty(X)$ is the multiplication operator. This operator will not map the identity on $\mathcal{C}^\infty$ onto the identity on $\mathrm{End}(E_k)$, but the hermiticity of operators is preserved under quantization.

As a side remark, note that the definition of a Berezin push and a Berezin-Toeplitz lift of operators readily extends to quantized supermanifolds. For the Berezin push, one can use the orthoprojector π_k defined in the paragraph after equation (4.34).

Because hermiticity of the quantum Laplace operator is the crucial property, we define a quantized version of the action functional (5.1) as

$$S_k[\phi_k] := \mathrm{tr}\big(\phi_k \hat{\Delta}_k(\phi_k) + V(\phi_k)\big) , \quad \phi_k \in \mathrm{End}(E_k) \ . \tag{5.5}$$

As the functional S_k lives on the finite dimensional space $\mathrm{End}(E_k)$, the corresponding functional integral

$$Z = \int_{\mathrm{End}(E_k)} \mathcal{D}[\phi_k] \ e^{-S_k[\phi_k]} \tag{5.6}$$

is a finite-dimensional integral and thus well-defined. This is what people refer to as fuzzy quantum scalar field theory [2, 10, 40], and besides providing a nice regularization procedure, using the quantized form (5.6), one can easily study the field theory (5.1) numerically on a computer [41, 42].

5.2 The $\mathcal{N} = (2,2)$ supersymmetric sigma model on $\mathbb{C}^{1|2}$

Considering the supersymmetric sigma model with $(2,2)$ supersymmetries on the superspace $\mathbb{C}^{1|2}$ is particular convenient as this space has the same volume form as $\mathbb{P}^{1|2}$ on one of the standard patches, for which a bosonic homogeneous coordinate, e.g. z^0, does not vanish. The reason for this is that $\mathbb{P}^{1|2}$ is a Calabi-Yau supermanifold.

Calabi-Yau supermanifolds. The spaces $\mathbb{P}^{n|n+1}$ come with a nowhere vanishing holomorphic volume form. Using the usual inhomogeneous coordinates $z_0^1, \ldots, z_0^n, \zeta_0^1, \ldots, \zeta_0^{n+1}$ on the patch $U_0 : z^0 \neq 0$ of $\mathbb{P}^{n|n+1}$, the superdeterminant in the Liouville measure (4.21) is just a constant, and thus

$$\Omega_{U_0}^{n|n+1,0|0} := \gamma\ z_0^1 \wedge \ldots \wedge dz_0^n d\zeta_0^1 \ldots d\zeta_0^{n+1} \tag{5.7}$$

can be extended to a non-vanishing globally holomorphic volume form. Here, $\gamma \in \mathbb{C}^*$ is an arbitrary nonvanishing constant. (Recall that $\mathrm{vol}(\mathbb{P}^{n|n+1}) = 0$, and therefore we cannot normalize by the space's natural volume, as one would usually do.) The Liouville measure is then given by $d\mu = \Omega^{n|n+1,0|0} \wedge \Omega^{0|0,n|n+1}$. It is evident that the Berezinian super line bundle of these spaces is trivial. Such spaces are referred to as *Calabi-Yau supermanifolds* in the literature. Note, however, that Yau's theorem doesn't hold without restrictions in the supercase, cf. [43]. In particular, the spaces $\mathbb{P}^{n|n+1}$ are not super Ricci-flat, i.e. the Ricci tensor

$$R_{IJ} := \frac{\partial^2 \log(\mathrm{sdet}(g))}{\partial Z^I \bar{Z}^J}\,, \tag{5.8}$$

where g is the super Kähler supermetric obtained from an arbitrary Kähler form ω, does not vanish. For our purposes, the existence of $\Omega^{n|n+1,0|0}$, or equivalently, triviality of the Berezinian super line bundle, will prove to be sufficient.

On $\mathbb{C}^{1|2}$, we introduce the supercovariant derivatives

$$D_{1,2} = \frac{\partial}{\partial \zeta^{1,2}} \pm \zeta^{1,2}\frac{\partial}{\partial z}\,, \quad \bar{D}_{1,2} = \frac{\partial}{\partial \bar{\zeta}^{1,2}} \pm \bar{\zeta}^{1,2}\frac{\partial}{\partial \bar{z}} \tag{5.9}$$

as well as the generators for supersymmetry transformations

$$Q_{1,2} = \frac{\partial}{\partial \zeta^{1,2}} \mp \zeta^{1,2}\frac{\partial}{\partial z}\,, \quad \bar{Q}_{1,2} = \frac{\partial}{\partial \bar{\zeta}^{1,2}} \mp \bar{\zeta}^{1,2}\frac{\partial}{\partial \bar{z}}\ . \tag{5.10}$$

Note that the relation to the usual chiral notation is as follows:

$$\zeta^+ = \frac{1}{\sqrt{2}}(\zeta^1 - \zeta^2)\,, \quad \zeta^- = \frac{1}{\sqrt{2}}(\zeta^1 + \zeta^2)\ . \tag{5.11}$$

The four basic superfields on $\mathbb{C}^{1|2}$ are then given by (cf. e.g. [44, 45])

$$\begin{array}{lll} D_2\Phi_{\mathrm{c}} = -D_1\Phi_{\mathrm{c}}\,, & \bar{D}_2\Phi_{\mathrm{c}} = -\bar{D}_1\Phi_{\mathrm{c}}\,, & D_+\Phi_{\mathrm{c}} = \bar{D}_+\Phi_{\mathrm{c}} = 0 \\ D_2\Phi_{\mathrm{ac}} = D_1\Phi_{\mathrm{ac}}\,, & \bar{D}_2\Phi_{\mathrm{ac}} = \bar{D}_1\Phi_{\mathrm{ac}}\,, & D_-\Phi_{\mathrm{ac}} = \bar{D}_-\Phi_{\mathrm{ac}} = 0 \\ D_2\Phi_{\mathrm{tc}} = -D_1\Phi_{\mathrm{tc}}\,, & \bar{D}_2\Phi_{\mathrm{tc}} = \bar{D}_1\Phi_{\mathrm{tc}}\,, & D_+\Phi_{\mathrm{tc}} = \bar{D}_-\Phi_{\mathrm{tc}} = 0 \\ D_2\Phi_{\mathrm{tac}} = D_1\Phi_{\mathrm{tac}}\,, & \bar{D}_2\Phi_{\mathrm{tac}} = -\bar{D}_1\Phi_{\mathrm{tac}}\,, & D_-\Phi_{\mathrm{tac}} = \bar{D}_+\Phi_{\mathrm{tac}} = 0 \end{array} \tag{5.12}$$

corresponding to *chiral* (c), *anti-chiral* (ac), *twisted chiral* (tc) and *twisted anti-chiral superfields* (tac), respectively. As we are working in Euclidean space, the notion of reality is slightly more subtle than in the Minkowski case. In particular, the ordinary complex conjugate of a chiral superfield is not an antichiral superfield, and one has to introduce a different real structure to obtain this result. However, twisted chiral superfields are indeed complex conjugate to twisted anti-chiral superfields and therefore we will restrict to them in most of the following. For convenience, we introduce the chiral and anti-chiral coordinates

$$\begin{aligned} z^+ &:= z+\zeta^1\zeta^2 = z+\zeta^+\zeta^- \,, \qquad & z^- &:= z-\zeta^1\zeta^2 = z-\zeta^+\zeta^- \,, \\ \bar z^+ &:= \bar z+\bar\zeta^1\bar\zeta^2 = \bar z+\bar\zeta^+\bar\zeta^- \,, \qquad & \bar z^- &:= \bar z-\bar\zeta^1\bar\zeta^2 = \bar z-\bar\zeta^+\bar\zeta^- \,, \end{aligned} \tag{5.13}$$

which satisfy $D_+z^- = D_-z^+ = 0$. From these fields, one can now construct an action using a real function $K(\Phi^1_{\rm c},\dots,\Phi^i_{\rm c},\Phi^1_{\rm c},\dots,\Phi^i_{\rm ac},\Phi^1_{\rm tc},\dots,\Phi^j_{\rm tc},\Phi^1_{\rm tac},\dots,\Phi^j_{\rm tac}) =: K(\Phi)$ as follows:

$$S = \int \mathrm{d}^2z\mathrm{d}^2\zeta^1\mathrm{d}^2\zeta^2 K(\Phi) \ . \tag{5.14}$$

When interpreting the superfields Φ as maps from the worldsheet $\mathbb{C}^{1|2}$ into a complex target manifold, one is led to regard the superfunction K as the Kähler potential of the target space if it only depends on chiral and anti-chiral superfields. We have

$$g_{ab} := \frac{\partial^2 K(\Phi)}{\partial\Phi^a_{\rm c}\partial\Phi^b_{\rm ac}} \,, \tag{5.15}$$

where g_{ab} is the target space supermetric. If twisted chiral superfields are included as well, there is an analogous relation to generalized complex geometry.

One can furthermore add superpotential terms of the form

$$\int \mathrm{d}^2z\mathrm{d}\zeta^-\mathrm{d}\zeta^+ W(\Phi_{\rm c}) \,, \qquad \int \mathrm{d}^2z\mathrm{d}\bar\zeta^-\mathrm{d}\zeta^+ \hat W(\Phi_{\rm tc}) \,, \tag{5.16}$$

which have to be accompanied by their complex conjugate. Here, W and $\hat W$ are polynomials in the chiral and twisted chiral superfields, restricted by renormalizability of the theory.

To be concise, let us now restrict[8] to a specific model which contains only twisted chiral superfields. (Recall that a sigma model containing only twisted chiral superfields is dual to one containing only untwisted ones). The superfield expansion of a twisted chiral superfield reads as

$$\begin{aligned} \Phi(z^+,\bar z^-,\zeta^+,\bar\zeta^-) &= \phi(z^+,\bar z^-) + \zeta^+\bar\psi^-(z^+,\bar z^-) + \bar\zeta^-\psi^+(z^+,\bar z^-) + \zeta^+\bar\zeta^- F(z^+,\bar z^-) \\ &= \phi(z,\bar z) + \zeta^+\zeta^-\partial_z\phi(z,\bar z) + \bar\zeta^+\bar\zeta^-\partial_{\bar z}\phi(z,\bar z) + \zeta^+\zeta^-\bar\zeta^+\bar\zeta^-\partial_z\partial_{\bar z}\phi(z,\bar z) \\ &\quad + \zeta^+\bar\psi^-(z,\bar z) + \zeta^+\bar\zeta^+\bar\zeta^-\partial_{\bar z}\bar\psi(z,\bar z) + \bar\zeta^-\psi^+(z,\bar z) \\ &\quad + \bar\zeta^-\zeta^+\zeta^-\partial_z\psi^+(z,\bar z) + \zeta^+\bar\zeta^- F(z,\bar z) \ . \end{aligned}$$

Putting

$$K(\Phi) = \bar\Phi_{\rm tc}\Phi_{\rm tc} \quad \text{and} \quad \hat W(\Phi_{\rm tc}) = m\Phi^2_{\rm tc} + \lambda\Phi^3_{\rm tc} \,, \tag{5.17}$$

[8] It should be stressed, that more general models could have been treated in principle.

we arrive at a sigma model with the component action

$$
\begin{aligned}
S = \int d^2 z \, \Big(& \bar{\phi}\partial_z \partial_{\bar{z}} \phi + \partial_{\bar{z}} \bar{\phi} \partial_z \phi + (\partial_z \partial_{\bar{z}} \bar{\phi}) \phi + F \bar{F} \\
& - \psi^- \partial_z \psi^+ + \bar{\psi}^+ \partial_{\bar{z}} \bar{\psi}^- - (\partial_{\bar{z}} \psi^-) \psi^+ + (\partial_z \bar{\psi}^+) \bar{\psi}^- \\
& + 2m^2 (\phi F - \bar{\psi}^- \psi^+) + 3\lambda (\phi^2 F - \phi \bar{\psi}^- \psi^+) + c.c. \Big) ,
\end{aligned} \tag{5.18}
$$

where all fields depend only on z (non-holomorphically, in general). After integrating out the auxiliary fields and integrating by parts, we arrive at the final form of the action

$$
\begin{aligned}
S = \int d^2 z \, \Big(& \bar{\phi}\partial_z \partial_{\bar{z}} \phi - \psi^- \partial_z \psi^+ + \bar{\psi}^+ \partial_{\bar{z}} \bar{\psi}^- - (\partial_{\bar{z}} \psi^-) \psi^+ + (\partial_z \bar{\psi}^+) \bar{\psi}^- \\
& + 3 |2m\phi + 3\lambda \phi^2|^2 \Big) .
\end{aligned} \tag{5.19}
$$

5.3 Regularization with Berezin-quantized $\mathbb{P}^{1|2}$

To regularize the theory (5.19), we would like to obtain a supersymmetric theory on $\mathbb{P}^{1|2}$, which, upon decompactification (or, equivalently, taking out a (super)point) turns into the supersymmetric sigma-model on $\mathbb{C}^{1|2}$. We can then translate the theory from $\mathbb{P}^{1|2}$ to Berezin-quantized $\mathbb{P}^{1|2}$ to obtain a finite quantum field theory.

Two issues remain to be clarified. The first one concerns the definition of chiral and twisted chiral superfields on $\mathbb{P}^{1|2}$ and the relation with supersymmetry transformations, while the second one is the integration over chiral and twisted chiral superspace.

As the space $\mathbb{P}^{1|2}$ is group theoretically given by the coset space $\mathrm{U}(2|2)/(\mathrm{U}(1|0) \times \mathrm{U}(1|2))$, its isometry group[9] is $\mathrm{U}(2|2)$. We will work at the level of the algebra of generators $u(2|2)$, and we will use the following Hermitian generators:

$$
(\sigma^{IJ})_{AB} = \varphi_{IJ}\delta_{IA}\delta_{JB} + \varphi_{JI}\delta_{IB}\delta_{JA} \quad \text{and} \quad (\rho^{IJ})_{AB} = i\varphi_{IJ}\delta_{IA}\delta_{JB} - i\varphi_{JI}\delta_{IB}\delta_{JA}, \tag{5.20}
$$

where $I, J, A, B \in 0, \ldots, 3$ and $\varphi_{IJ} = e^{\pi i/2\tilde{I} - \pi i/2\tilde{J}}$ is a phase factor necessary to guarantee that our norm of vectors in $\mathbb{C}^{2|2}$ is invariant. In particular, σ^{IJ} and ρ^{IJ} generate space-time rotations for $I \leq 1, J \leq 1$, R-symmetry rotations for $I \geq 2, J \geq 2$ and supersymmetry transformations in all other cases.

The representation R of these generators acting on superfunctions on $\mathbb{C}^{2|2}$ is given by the differential operators

$$
R(\sigma^{IJ}) = Z^I \sigma^{IJ} \frac{\partial}{\partial Z^J} - \bar{Z}^I \sigma^{IJ} \frac{\partial}{\partial \bar{Z}^J} \quad \text{and} \quad R(\rho^{IJ}) = Z^I \rho^{IJ} \frac{\partial}{\partial Z^J} + \bar{Z}^I \rho^{IJ} \frac{\partial}{\partial \bar{Z}^J} , \tag{5.21}
$$

where $(Z^I) = (z^0, z^1, \zeta^1, \zeta^2)$ are the coordinates on $\mathbb{C}^{2|2}$. Note that $|Z|^2$ is invariant as expected. To obtain the corresponding action on $\mathbb{P}^{1|2}$, we have these symmetries act on a certain patch U on the inhomogeneous coordinates Z_0^I. Consider again the patch U_0 for which $z^0 \neq 0$, then we have in addition to the generators

$$
R_0(\sigma^{IJ}) = Z_0^I \sigma^{IJ} \frac{\partial}{\partial Z_0^J} - \bar{Z}_0^I \sigma^{IJ} \frac{\partial}{\partial \bar{Z}_0^J} \quad \text{and} \quad R_0(\rho^{IJ}) = Z_0^I \rho^{IJ} \frac{\partial}{\partial Z_0^J} + \bar{Z}_0^I \rho^{IJ} \frac{\partial}{\partial \bar{Z}_0^J} \quad , \; I, J \geq 1
$$

[9]Here, we choose to use the full unitary supergroup to avoid discussing the projective subgroup $PSU(2|2)$.

the generators

$$R_0(\sigma^{00}) = 0\,, \quad R_0(\sigma^{0I}) = \frac{\partial}{\partial Z_0^I} - Z_0^I\mathcal{E} - \frac{\partial}{\partial \bar{Z}_0^I} + \bar{Z}_0^I\bar{\mathcal{E}}\,,$$
$$R_0(\rho^{0I}) = i\left(\frac{\partial}{\partial Z_0^I} + Z_0^I\mathcal{E} + \frac{\partial}{\partial \bar{Z}_0^I} + \bar{Z}_0^I\bar{\mathcal{E}}\right)$$

for $i \leq 1$ and $\mathcal{E} := z_0^1\partial_{z_0^1} + \zeta_0^1\partial_{\zeta_0^1} + \zeta_0^2\partial_{\zeta_0^2}$. Note that the expression $|Z_0|^2 := 1 + z\bar{z} + i\zeta^1\bar{\zeta}^1 + i\zeta^2\bar{\zeta}^2$ is only invariant under transformations $R_0(\sigma^{IJ})$ with $I, J \geq 1$. When decompactifying $\mathbb{P}^{1|2}$ to $\mathbb{C}^{1|2}$, the Euler operators $\mathcal{E}$ vanish, and we can thus identify the differential operators D_1 and D_2 with the generators according to

$$\begin{aligned} D_1 = D_1^{R_0} &= \frac{1}{2}(R_0(\sigma^{02}) - iR_0(\rho^{02}) + R_0(\sigma^{21}) - iR_0(\rho^{21}))\,, \\ D_2 = D_2^{R_0} &= \frac{1}{2}(R_0(\sigma^{03}) - iR_0(\rho^{03}) - R_0(\sigma^{31}) + iR_0(\rho^{31}))\,. \end{aligned} \tag{5.22}$$

Using the corresponding differential operators $D_{1,2}^R$ in the representation R, we have an action on monomials in the homogeneous coordinates, which preserves their bi-degree. Recall that superfunctions on $\mathbb{P}^{1|2}$ are written in terms of basis superfunctions

$$\frac{Z^{I_1}\ldots Z^{I_k}\bar{Z}^{J_1}\ldots\bar{Z}^{J_k}}{|Z|^{2k}}\,, \tag{5.23}$$

and the action of $D_{1,2}^R$ on these superfunctions is given as the action of the differential operators in coordinates of $\mathbb{C}^{2|2}$ on the numerator. (The denominator is invariant under $u(2|2)$-transformations.) This allows us to define all the chiral superfields as above in (5.12). Note that a superfield of any of the possible chiralities will transform into a non-chiral superfield under arbitrary $u(2|2)$ supersymmetry transformations, as $D_{1,2}^R$ does not anticommute with general supersymmetry transformations. However, the number of independent component fields remains evidently the same and is merely reshuffled in the field expansion. We will come back to this point later.

The second issue is the integration over chiral and anti-chiral superspace to allow for the inclusion of a non-trivial superpotential. As the only invariant measure available is the full integral over superspace, we have to insert a superfunction, which takes care of the antichiral part:

$$\int d^2z d\zeta^+ d\bar{\zeta}^- \;\rightarrow\; \int d\mu([Z])\frac{\bar{\zeta}^+\zeta^-}{|Z|^2}\,, \tag{5.24}$$

where $d\mu([Z])$ is again the super Liouville measure on $\mathbb{P}^{1|2}$. Note that indeed $\frac{\bar{\zeta}^+\zeta^-}{|Z|^2} \in \mathcal{C}^\infty(\mathbb{P}^{1|2})$. After integrating out the auxiliary fields, the factor $\frac{1}{|Z|^2}$ will produce a factor of $\frac{1}{|z|^4}$ in front of potential terms, the usual Liouville measure on $\mathbb{P}^1$. This will produce the correct planar limit, after decompactifying $\mathbb{P}^1$ to $\mathbb{C}^1$.

To regularize this model on $\mathbb{P}^{1|2}$ by Berezin-quantizing the worldsheet as $(\mathbb{P}^{1|2}, E := O(k))$, we need to translate all the above machinery to the quantum situation. First, superfields are now elements of End_k, and this space is spanned by the operators

$$\hat{A}^\dagger_{I_1}\ldots\hat{A}^\dagger_{I_k}|0\rangle\langle 0|\hat{A}_{J_1}\ldots\hat{A}_{J_k}\,, \tag{5.25}$$

cf. section 5. The $u(2|2)$ invariant integral is given by the supertrace

$$\int_{\mathbb{P}^{1|2}} d\mu(z)\sigma(\hat{f}) = \frac{\mathrm{vol}'(\mathbb{P}^{1|2})}{b_k^0 + b_k^1}\mathrm{str}(\hat{f}) \tag{5.26}$$

and the representation $\hat{R}$ of the generators σ^{IJ} and ρ^{IJ} on End_k is the usual Schwinger representation

$$\hat{R}(\sigma^{IJ})(\hat{f}) = [\![\hat{A}_A^\dagger \sigma_{AB}^{IJ}\hat{A}_B, \hat{f}]\!]\,, \quad \hat{R}(\rho^{IJ})(\hat{f}) = [\![\hat{A}_A^\dagger \rho_{AB}^{IJ}\hat{A}_B, \hat{f}]\!]\ . \tag{5.27}$$

While the definition of twisted chiral and twisted anti-chiral superfields in this manner goes over into ordinary twisted chiral and twisted anti-chiral superfields upon decompactification, one might argue that it is still too restrictive. Having in mind that D_1 and D_2 act only holomorphically on the fields, one could restrict the actions of $\hat{R}(\sigma^{IJ})$ in the definition of twisted chiral superfields to their left-actions, which amounts to a holomorphic action on the corresponding fields. This point is quite subtle and requires certainly further scrutiny.

The integral over chiral superspace can now be performed in two different ways. Either, we multiply the operator to be integrated over chiral superspace with the operator corresponding to the superfunction $\frac{\bar{\zeta}^+\zeta^-}{|Z|^2}$ and integrate via the supertrace on End_k, or we add the corresponding creation and annihilation operators to all superfunctions by insertion and integrate by taking the supertrace over End_{k+1}. Here, we will choose to work with the former procedure. Putting everything together, we have the following action:

$$S = \frac{\mathrm{vol}'(\mathbb{P}^{1|2})}{b_k^0 + b_k^1}\,\mathrm{str}\Big(\hat{\Phi}_{\mathrm{tc}}^\dagger\hat{\Phi}_{\mathrm{tc}} + \hat{\Psi}\hat{W}(\hat{\Phi}_{\mathrm{tc}}) + \hat{\Psi}^\dagger\hat{W}^\dagger(\hat{\Phi}_{\mathrm{tc}}^\dagger)\Big)\,, \tag{5.28}$$

where $\hat{W}(\hat{\Phi}_{\mathrm{ch}})$ is again a polynomial in its argument $\hat{\Phi}_{\mathrm{ch}}$ and

$$\hat{\Psi} := \alpha_+^\dagger \hat{A}_{I_1}^\dagger \dots \hat{A}_{I_{k-1}}^\dagger |0\rangle\langle 0|\hat{A}_{I_{k-1}}\dots\hat{A}_{I_1}\alpha_-\ . \tag{5.29}$$

The superfunctional integral has now to be taken over all operators corresponding to twisted chiral fields

$$Z \ = \ \int \mathcal{D}\Phi_{\mathrm{tc}}\exp(-S)\ ; \tag{5.30}$$

it is a finite integral and thus provides a regularization of the $\mathcal{N}=(2,2)$ supersymmetric sigma model in two dimensions in the usual sense of fuzzy geometry.

Remarks. The original sigma-model on $\mathbb{C}^{1|2}$ was invariant under 4 (real) supercharges: $Q_\pm, \bar{Q}_\pm$. The algebra of isometries of $\mathbb{P}^{1|2}$ contains, however, 8 odd generators. This shows up in the fact that the definition of a twisted chiral superfield is not invariant[10] under half of the $u(2|2)$ generators. Without superpotential term, the global symmetry group of the action (5.28) is indeed $\mathrm{U}(2|2)$, and the supersymmetry transformations modifying twisted chiral superfields merely reshuffle the component fields. This invariance is easily seen as the D-term $\mathrm{str}(\hat{\Phi}_{\mathrm{tc}}^\dagger\hat{\Phi}_{\mathrm{tc}})$ is evidently invariant under transformations $\hat{\Phi}_{\mathrm{tc}} \to \hat{U}\hat{\Phi}_{\mathrm{tc}}\hat{U}^\dagger$. A superpotential term, however, breaks the supersymmetry of the model down to the same as the one on $\mathbb{C}^{1|2}$, which we set out to regularize in the first place, and this was in fact to be expected.

[10] This is clear from group theoretic considerations.

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5.4 Comments on the topological twist

In a more general context, the above mentioned sigma model can be defined on an arbitrary Riemann surface with canonical bundle K. The associated super Riemann surface is a split supermanifold which is the total space of the (real) rank 4 vector bundle

$$(\Pi K^{1/2} \oplus \Pi \bar{K}^{1/2}) \oplus (\Pi K^{1/2} \oplus \Pi \bar{K}^{1/2}) \ . \tag{5.31}$$

In the language of [18] and section 2, this is a superspace $(X, \mathcal{A})$ such that $X_{\rm red}$ is a Riemann surface and $\hat{\mathcal{A}}$ isomorphic to the sheaf of supersections of the vector bundle (5.31) with $\mathcal{A}$ globally isomorphic to $\wedge^n_{\mathcal{A}_{\rm red}}\hat{\mathcal{A}}$. In our example, $K = O(2)$ and $\zeta^+, \bar{\zeta}^+$ are supersections of the first two super line bundles, while $\zeta^-, \bar{\zeta}^-$ are supersections of the second two. Applying a topological twist (see e.g. [44]), we deform this geometry to

$$(\Pi O \oplus \Pi K) \oplus (\Pi O \oplus \Pi \bar{K}) \ . \tag{5.32}$$

In our example, the resulting space would be the weighted superprojective space $W\mathbb{P}^{1|2}(1,1|0,2)$, which is the total space of the vector bundle $O \oplus \Pi O(2)$ over $\mathbb{P}^1$.

While on flat space, this twist corresponds to a mere rewriting, on curved space, the twist allows for defining supersymmetric models on non-spin manifolds and avoids the introduction of spinors altogether. In particular, the Graßmann coordinates parametrizing the trivial super line bundle give rise to supercharges which carry Lorentz spin 0 and are thus invariant under space-time rotations. This guarantees the preservation of a certain amount of supersymmetry.

The definition of chiral and twisted chiral fields on this geometry proceeds as before, and following the procedure of the untwisted case, one eventually arrives at a topologically twisted sigma model on quantized $W\mathbb{P}^{1|2}(1,1|0,2)$.

6 Summary and directions for further research

In this paper, we defined generalized Berezin and Berezin-Toeplitz quantization of Hodge supermanifolds. A prerequisite for this quantization was the given extension of the Rawnsley coherent states to the case of supermanifolds. Explicitly, we constructed the quantization of both affine and projective superspaces. Eventually, we showed how one can employ such quantized supermanifolds as supersymmetry-preserving regulators of quantum field theories; we proposed definitions of ordinary and twisted $\mathcal{N} = (2,2)$ supersymmetric sigma models on the compactified superspace $\mathbb{P}^{1|2}$.

Taking our results as a starting point, one has a number of potentially interesting directions for future research. Clearly, it would be desirable to expose more supersymmetric field theories admitting a regularization by Berezin-quantized Hodge supermanifolds. One is evidently restricted to such theories which allow for a superfield formulation. However it is unclear, whether one is limited to using the quantizations of Calabi-Yau supermanifolds in regularizing supersymmetric field theories on flat superspace. Moreover, an extension to supersymmetric gauge theories is desirable, having in mind the ultimate aim of the minimal supersymmetric standard model regularized on a fuzzy superspace. Also, one would expect that the topological twist plays a crucial role in regularizing supersymmetric field theories using more general quantized Hodge supermanifolds as it allows for working without spinors.

A natural question with respect to the nonlinear sigma models regularized above would be whether mirror symmetry holds after regularization. This would require a more general analysis of $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma models on Calabi-Yau manifolds but this "fuzzy mirror symmetry" would be useful in the associated $\mathcal{N} = 2$ superconformal algebra calculations and thus it might help with numerical studies of mirror symmetry.

Numerical studies[11] of the models proposed above and their generalizations can be readily performed, and the behavior of the regulated models should be compared to the conventional knowledge of supersymmetric field theories. Note also that here, one is analyzing a supermatrix model, and the application of matrix model techniques to these regulated supersymmetric field theories in the spirit of [47] might yield more interesting results than in the non-supersymmetric case.

More formally, it seems to be a mere technicality to extend the relation between geometric quantization and formal deformation quantization using Berezin-Toeplitz quantization to the case of supermanifolds. Eventually, one might wish to extend the known relationship between quantizable Hermitian symmetric spaces and the Toeplitz quantization procedure [16] to the case of supermanifolds.

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